MATH3290 Mathematical Modeling 2024/2025
Answers to Assignment 1

1. Let a_n be the amount owe at the end of the n-th month. Therefore, the corresponding difference equation can be written as

$$\begin{cases} \Delta a_n = 0.015 a_n - 50, \\ a_0 = 500. \end{cases}$$

By the difference equation, we have

$$a_n = (a_0 - \frac{10000}{3}) \times 1.015^n + \frac{10000}{3}.$$

Let $a_n = 0$. We can get $n \approx 11$. The balance will be paid off after 11 months.

2. (a) The model $W^1 = cg^3$ assumes that weight is directly proportional to the cube of girth while independent of length.
The model $W^2 = kgl^2$ assumes that weight is directly proportional to girth and also to the square of length.

(b) Fitting each model to the data by the least-squares criterion, we need to minimize

$$\begin{aligned} S^1(c) &= \sum_{i=1}^{m} (W_i - cg_i^3)^2 \\ S^2(k) &= \sum_{i=1}^{m} (W_i - kg_i l_i^2)^2 \end{aligned}$$

To find the minimum, we let

$$\begin{aligned} 0 &= \frac{dS^1}{dc}(c) = \sum_{i=1}^{m} (2cg_i^6 - 2W_i g_i^3) \\ 0 &= \frac{dS^2}{dk}(k) = \sum_{i=1}^{m} (2kg_i^2 l_i^4 - 2W_i g_i l_i^2) \end{aligned}$$

Then with the data, we can solve the equations and get $c = 0.0276$ and $k = 0.0126$. Consider $S^1(c)$ and $S^2(k)$,

$$S^1(c = 0.0276) = 54.2563 > 3.3885 = S^2(k = 0.0126)$$

we evaluate the model $W^2 = kgl^2$ is better.

3. (a) Formulating the system of difference equations, we have the following dynamical system:

$$\begin{aligned} C_1^{n+1} &= 0.3C_1^n + 0.3C_2^n + 0.4C_3^n \\ C_2^{n+1} &= 0.2C_1^n + 0.5C_2^n + 0.2C_3^n \\ C_3^{n+1} &= 0.5C_1^n + 0.2C_2^n + 0.4C_3^n \end{aligned}$$

Then $\begin{pmatrix} C_1^{n+1} \\ C_2^{n+1} \\ C_3^{n+1} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.5 & 0.2 \\ 0.5 & 0.2 & 0.4 \end{pmatrix} \begin{pmatrix} C_1^n \\ C_2^n \\ C_3^n \end{pmatrix}$. Denote this equations as $C_{n+1} = AC_n$.

(b) Suppose (C_1^*, C_2^*, C_3^*) are the equilibrium values of this system, then we must have $C_1^* = C_1^{n+1} = C_1^n, C_2^* = C_2^{n+1} = C_2^n, C_3^* = C_3^{n+1} = C_3^n$. Substituting into the dynamical system yields

$$\begin{aligned} 0.7C_1^* &= 0.3C_2^* + 0.4C_3^* \\ 0.5C_2^* &= 0.2C_1^* + 0.2C_3^* \\ 0.6C_3^* &= 0.5C_1^* + 0.2C_2^*. \end{aligned}$$

There are an infinite number of solutions to this system of equations. Suppose $C_3^* = 1$, we find the dynamical system is satisfied when $C_1^* = 0.8966, C_2^* = 0.7586$. Keep this proportion and let C_1^*, C_2^*, C_3^* satisfy $C_1^* + C_2^* + C_3^* = 1$, we can get the equilibrium point $C^* = (0.3377, 0.2857, 0.3766)$.

To determine the stability of this equilibrium point, first we calculate the eigenvalues of A. They are $\lambda_1 = 1, \lambda_2 = -0.1, \lambda_3 = 0.3$. And the corrsponding eigenvectors are

$$X_1 = \begin{pmatrix} -0.5812 \\ -0.4918 \\ -0.6483 \end{pmatrix}, X_2 = \begin{pmatrix} -0.7071 \\ 0.0000 \\ 0.7071 \end{pmatrix}, X_3 = \begin{pmatrix} 0.1961 \\ -0.7845 \\ 0.5883 \end{pmatrix}.$$

Since X_1, X_2, X_3 are linearly independent, then C_0 can be written as

$$C_0 = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3.$$

Then

$$\begin{aligned} C_n &= A^n C_0 = A^n(\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3) \\ &= A^{n-1}(\alpha_1 \lambda_1 X_1 + \alpha_2 \lambda_2 X_2 + \alpha_3 \lambda_3 X_3) \\ &\cdots \\ &= \alpha_1 \lambda_1^n X_1 + \alpha_2 \lambda_2^n X_2 + \alpha_3 \lambda_3^n X_3. \end{aligned}$$

Since $|\lambda_2| < 1, |\lambda_3| < 1$, then

$$\lim_{n\to+\infty} \lambda_2^n = 0, \lim_{n\to+\infty} \lambda_3^n = 0.$$

Hence,

$$\lim_{n\to+\infty} C_n = \alpha_1 X_1.$$

So this equilibrium point is stable and independent of the initial condition.

(c) Use the equations in (a), it's easy to compute C_1^5, C_2^5, C_3^5 for each case. The tendency in each case can be seen in Figure 1-3. It's esay to see that the long term behaviour does not sensitive to the initial condition.

Percentage	C_1^5	C_2^5	C_3^5
Case A	0.3378	0.2850	0.3771
Case B	0.3375	0.2862	0.3762
Case C	0.3377	0.2855	0.3768

Table 1: Data set for Problem 3.

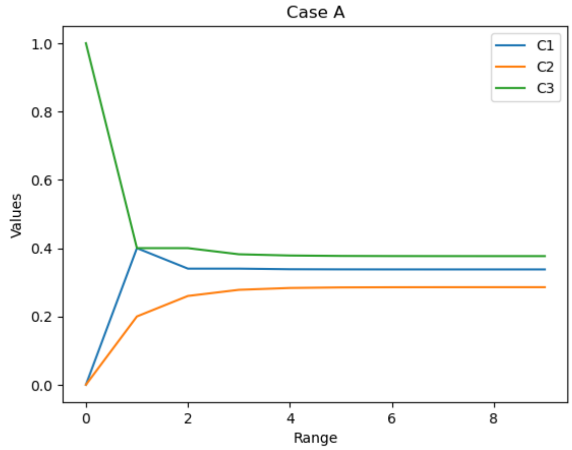

Figure 1: Case A of Problem 3(c)

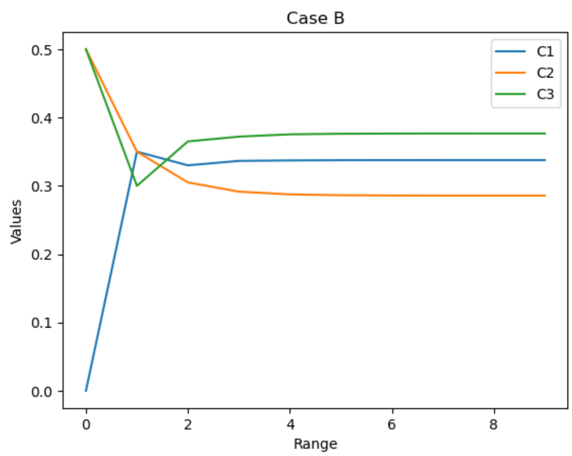

Figure 2: Case B of Problem 3(c)

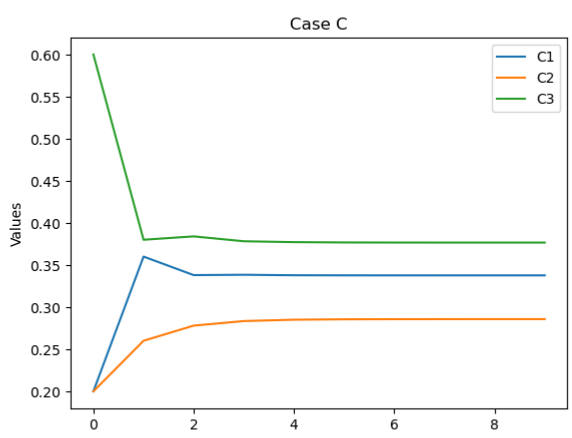

Figure 3: Case C of Problem 3(c)