Solution 3

1. Let f be a 2π-periodic function integrable on $[-\pi, \pi]$ whose Fourier series is the zero function. Show that

   (a)
   $$\int_{-\pi}^{\pi} f(x)g(x)\,dx = 0 \ ,$$
   for all continuous, 2π-periodic functions g.

   (b)
   $$\int_{-\pi}^{\pi} f(x)s(x)\,dx = 0 \ ,$$
   for all step functions s, and

   (c) Deduce that $f = 0$ almost everywhere.

   Solution

   (a) The vanishing of all Fourier coefficients means that
   $$\int_{-\pi}^{\pi} f(x)T(x)\,dx = 0,$$
   for all trigonometric polynomial T. By Weierstrass Approximation theorem every continuous, 2π-periodic function can be approximated by trigonometric polynomials in sup-norm. It follows that
   $$\int_{-\pi}^{\pi} f(x)g(x)\,dx = 0,$$
   for all g.

   (b) It is easy to see that we can approximate a step function s by a continuous function g. More precisely, given $\varepsilon > 0$, there is some continuous g such that
   $$\int_{-\pi}^{\pi} |s(x) - g(x)|\,dx < \varepsilon.$$
   Using this observation, one can show that
   $$\int_{-\pi}^{\pi} f(x)s(x)\,dx = 0$$
   for all step functions.

   (c) From (2) we deduce that $\int_{-\pi}^{\pi} f^2(x)\,dx = 0$, hence $f(x) = 0$ almost everywhere.

2. Show that the "Fourier map" $f \mapsto \hat{f}(n) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ satisfies $\hat{f} = \hat{g}$ if and only if $f = g$ almost everywhere.

   Solution It follows from the previous problem by replacing f by $f - g$.

3. Prove Hölder's Inequality in vector form: For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $p > 1$ and q conjugate to p,

$$|\mathbf{x} \cdot \mathbf{y}| \le \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \left(\sum_{j=1}^{n} |y_j|^q \right)^{1/q} .$$

You may prove it directly or deduce it from its integral form by choosing suitable functions f anf g.

Solution. Dividing $[0,1]$ equally into n many subintervals I_j and set $f(x) = x_j, g(x) = y_j$, for $x \in (x_j, x_{j+1}]$, Hölder's inequality for vectors follows from the same inequality for f and g.

4. Prove Minkowski's Inequality in vector form: For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $p > 1$,

$$\|\mathbf{x} + \mathbf{y}\|_p \le \|\mathbf{x}\|_p + \|\mathbf{y}\|_p .$$

You may prove it directly or deduce it from its integral form by choosing suitable functions f anf g.

Solution. Same as in the previous problem.

5. Prove the generalized Hölder's Inequality: For $f_1, f_2, \cdots, f_n \in R[a,b]$,

$$\int_a^b |f_1 f_2 \cdots f_n| dx \le \left(\int_a^b |f_1|^{p_1} \right)^{1/p_1} \left(\int_a^b |f_2|^{p_2} \right)^{1/p_2} \cdots \left(\int_a^b |f_n|^{p_n} \right)^{1/p_n} ,$$

where

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1, \quad p_1, p_2, \cdots, p_n > 1 .$$

Solution. Induction on n. $n = 2$ is the original Hölder, so it holds. Let

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{n+1}} = 1 .$$

First, using the original Hölder, we have

$$\int_a^b |f_1 f_2 \cdots f_{n+1}| \, dx \le \left(\int_a^b |f_1|^{p_1} \, dx \right)^{1/p_1} \left(\int_a^b |f_2 \cdots f_{n+1}|^q \, dx \right)^{1/q} ,$$

where q is conjugate to p_1. It is easy to see

$$1 = \frac{q}{p_2} + \cdots + \frac{q}{p_{n+1}} .$$

By induction hypothesis,

$$\int_a^b |f_2^q \cdots f_n^q| \, dx \le \left(\int_a^b |f_2|^{p_2} \, dx \right)^{1/p_2} \cdots \left(\int_a^b |f_{n+1}|^{p_{n+1}} \, dx \right)^{1/p_{n+1}} ,$$

done.

6. Show that for $1 \le p < r \le \infty$,

(a)

$$\|\mathbf{x}\|_p \le n^{\frac{1}{p} - \frac{1}{r}} \|\mathbf{x}\|_r ,$$

(b)

$$\|\mathbf{x}\|_r \leq n^{\frac{1}{r}} \|\mathbf{x}\|_p.$$

Solution. (a)

$$\begin{aligned} \|\mathbf{x}\|_p^p &= \sum |x_j|^p \\ &\leq \left(\sum |x_j|^{p\frac{r}{p}}\right)^{\frac{p}{r}} \left(\sum 1^{\frac{r}{r-p}}\right)^{\frac{r-p}{r}} \\ &= n^{\frac{r-p}{r}} \|\mathbf{x}\|_r^p \end{aligned}$$

so

$$\|\mathbf{x}\|_p \leq n^{\frac{1}{p}-\frac{1}{r}} \|\mathbf{x}\|_r \ .$$

(b) First of all, $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_p$. Then,

$$\begin{aligned} \|\mathbf{x}\|_r &\leq (n\|\mathbf{x}\|_\infty^r)^{\frac{1}{r}} \\ &\leq n^{\frac{1}{r}} \|\mathbf{x}\|_\infty \\ &\leq n^{\frac{1}{r}} \|\mathbf{x}\|_p \ . \end{aligned}$$

7. Establish the inequality, for $f \in R[a,b]$, $\|f\|_p \leq C\|f\|_r$ when $1 \leq p < r$ for some constant C.

   Solution By Holder's Inequality,

$$\int_a^b |f|^p \leq \left(\int_a^b 1\,dx\right)^{1-p/r} \left(\int_a^b |f|^{p\frac{r}{p}}\,dx\right)^{p/r} \leq C\|f\|_r^p \ ,$$

   where

$$C = (b-a)^{\frac{1}{p}-\frac{1}{r}} \ .$$

8. Show that there is no constant C such that $\|f\|_2 \leq C\|f\|_1$ for all $f \in C[0,1]$.

   Solution Consider the sequence

$$f_n(x) = \begin{cases} -n^3x + n, & x \in [0, 1/n^2], \\ 0, & x \in (1/n^2, 1]. \end{cases}$$

   We have $\|f_n\|_1 = 1/(2n) \to 0$ as $n \to \infty$, but $\|f_n\|_2 = 1/3$ for all n. Hence, it is impossible to have some C satisfying $\|f\|_2 \leq C\|f\|_1$ for all f.

   Note. In general, it is impossible to find a constant C such that $\|f\|_r \leq C\|f\|_p, p < r$, for all f.

9. Show that $\|\cdot\|_p$ is no longer a norm on $\mathbb{R}^n$ for $p \in (0,1)$.

   Solution Again (N3) is bad. Consider two functions $f = \chi_{[0,1/2]}$ and $g = \chi_{[1/2,1]}$. We have $\|f+g\|_p = 1$ but $\|f\|_p = \|g\|_p = 2^{-1/p}$, so $\|f+g\|_p > \|f\|_p + \|g\|_p$. Although f and g are not continuous, we could find continuous approximations to these functions with the same effect.

10. In a metric space (X,d), its metric ball is the set $\{y \in X : \ d(y,x) < r\}$ where x is the center and r the radius of the ball. May denote it by $B_r(x)$. Draw the unit metric balls centered at the origin with respect to the metrics d_2, d_∞ and d_1 on $\mathbb{R}^2$.

Solution. The unit ball $B_1^2(0)$ is the standard one, the unit ball in d_∞-metric consists of points (x, y) either $|x|$ or $|y|$ is equal to 1 and $|x|, |y| \leq 1$, so $B_1^\infty(0)$ is the unit square. The unit ball $B_1^1(0)$ consists of points (x, y) satisfying $|x|+|y| \leq 1$, so the boundary is described by the curves $x + y = 1, x, y \geq 0,\ x - y = 1, x \geq 0, y \leq 0, -x + y = 1, x \leq 0, y \geq 0$, and $-x - y = 1, x, y \leq 0$. The result is the tilted square with vertices at $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$.