# Solution 2

1. A bounded function $f$ on $[a,b]$ is said to be locally Lipschitz continuous at $x \in [a,b]$ if there exist some $L$ and $\delta$ such that

$$|f(y)-f(x)| \le L|x-y| \ , \quad \forall y \in (x-\delta, x+\delta).$$

Show that $f$ is Lipschitz continuous at $x$.

**Solution.** For $y$ lying outside $(x-\delta, x+\delta)$, $|y-x| \ge \delta$. Therefore,

$$|f(y)-f(x)| = \frac{|f(y)-f(x)|}{|y-x|}\ |y-x| \le \frac{2\|f\|_\infty}{\delta}|y-x| \ .$$

Hence.

$$|f(y)-f(x)| \le L'(|y-x|, \quad \forall y \ , \quad L' = \max\{L, 2\|f\|_\infty/\delta\} \ .$$

Note This problem shows that like continuity Lipschitz continuity is also a local property, although in our definition it seems a global one.

2. Let $f$ be a function defined on $(a,b)$ and $x_0 \in (a,b)$.

   (a) Show that $f$ is Lipschitz continuous at $x_0$ if its left and right derivatives exist at $x_0$.

   (b) Construct a function Lipschitz continuous at $x_0$ whose one sided derivatives do not exist.

**Solution.** (a) Let $\alpha = f'_+(x_0)$ and $\beta = f'_-(x_0)$. For $\varepsilon = 1 > 0$, there exists $\delta_1$ such that

$$\left|\frac{f(x+z)-f(x)}{z} - \alpha\right| < 1 \ ,$$

for $0 < z < \delta_1$. It follows that

$$|f(x+z)-f(x)| \le |f(x+z)-f(x)-\alpha z| + |\alpha z| \le (1+|\alpha|)|z| \ .$$

Similarly,

$$|f(x+z)-f(x)| \le (1+|\beta|)|z| \ , \quad z \in (-\delta_2, 0) \ .$$

We conclude that $|f(x+z)-f(x)| \le (1+\gamma)|z|, \quad z \in (-\delta,\delta),\ \delta = \min\{\delta_1,\delta_2\}\ , \gamma = \max\{|\alpha|,|\beta|\}$. By Problem 1, it is Lipschitz continuous at $x_0$.

(b) The function $f(x) = x\sin\frac{1}{x}$ $(x \neq 0)$ and $= 0$ at $x = 0$. It is Lipschitz continuous at $x_0 = 0$ with $L = 1$ but both one-sided derivatives do not exist.

3. Provide a proof of Theorem 1.6.

**Solution** See Notes.

4. (a) Show that the Fourier series of the function $\cos tx$, $x \in [-\pi,\pi]$ where $t$ is not an integer is given by

$$\frac{\pi\cos tx}{\sin t\pi} = \frac{1}{t} + \sum_{n=1}^\infty \frac{2t}{t^2-n^2}(-1)^n\cos nx, \quad x \in [-\pi.\pi].$$

(b) Deduce that for $t \in (0,1)$,

$$\log \sin t\pi = \log t\pi + \sum_{n=1}^{\infty} \log\left(1 - \frac{t^2}{n^2}\right).$$

(c) Conclude that

$$\frac{\sin t\pi}{\pi t} = \prod_{n=1}^{\infty}\left(1 - \frac{t^2}{n^2}\right), \quad t \in (0,1).$$

**Solution** Using integration by parts, one has

$$f(x) = \frac{\pi \cos tx}{\sin t\pi} \sim \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2}(-1)^n \cos nx.$$

Since $f$ is smooth on $(-\pi, \pi)$ and $f(\pi^-) = f(-\pi^+)$, one has, by Theorem 1.6,

$$\frac{\pi \cos tx}{\sin t\pi} = \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2}(-1)^n \cos nx, \quad x \in [-\pi.\pi].$$

For $t \in (0,1)$, let

$$g(t) = \log \frac{\sin t\pi}{t\pi},$$
$$h(t) = \sum_{n=1}^{\infty} \log\left(1 - \frac{t^2}{n^2}\right).$$

Note that $h$ is well-defined since

$$|\log(1 - t^2/n^2)| \le 2t^2/n^2 \le 2/n^2, \quad \text{for } n \ge 2, t \in (0,1).$$

Since $(\log(1 - t^2/n^2))' = 2t/(t^2 - n^2)$ and $\sum_{n=1}^{\infty} 2t/(t^2 - n^2)$ converges uniformly on any $[a,b] \subset (0,1)$, $h'(t)$ is obtained by termwise differentiation and hence

$$h'(t) = \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2}, \quad \text{for any } t \in (0,1).$$

Since $\lim\limits_{t \to 0} \frac{\sin t}{t} = 1$, it is clear that

$$g(0^+) = 0 = h(0^+).$$

By (a), one has

$$g'(t) = f(\pi) - \frac{1}{t} = \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2} = h'(t),$$

Hence

$$h(t) = g(t), \quad t \in (0,1).$$

One then has (b) and (c).

5. Can you find a cosine series which converges uniformly to the sine function on $[0, \pi]$? If yes, find one.

   **Solution.** Yes, extend the sine function on $[0, \pi]$ to $|\sin x|$, an even, $2\pi$-periodic function. Since it is continuous, piecewise $C^1$, its cosine series converges uniformly to this extended function. In particular, this cosine series converges uniformly to $\sin x$ on $[0, \pi]$.

6. A sequence $\{a_n\}, n \geq 0$, is said to converge to $a$ in mean if

$$\frac{a_0 + a_1 + \cdots + a_n}{n+1} \to a\ , \quad n \to \infty\ .$$

   (a) Show that $\{a_n\}$ converges to $a$ in mean if $\{a_n\}$ converges to $a$.

   (b) Give a divergent sequence which converges in mean.

   **Solution**. (a) For $\varepsilon > 0$, there is some $n_0$ such that $|a_n - a| < \varepsilon$ for all $n > n_0$. Now

$$\begin{aligned}
\left|\frac{a_0 + \cdots + a_n}{n+1} - a\right| &= \left|\frac{a_0 + \cdots + a_{n_0}}{n+1} + \frac{a_{n_0+1} + \cdots + a_n}{n+1} - a\right| \\
&= \left|\frac{a_0 + \cdots + a_{n_0}}{n+1} + \frac{(a_{n_0+1} - a) + \cdots + (a_n - a)}{n+1} - \frac{n_0+1}{n+1}a\right| \\
&\leq \frac{n - n_0}{n+1}\varepsilon + \frac{a_0 + \cdots + a_{n_0}}{n+1} + \frac{n_0+1}{n+1}a \\
&\leq 2\varepsilon,
\end{aligned}$$

   after taking $n \geq n_1$ for a much larger $n_1$.

   (b) Consider the sequence $\{(-1)^n\}$.

7. Let $D_n$ be the Dirichlet kernel and define the Fejer kernel to be $F_n(x) = \dfrac{1}{n+1}\sum_{k=0}^n D_k(x)$.

   (a) Show that

$$F_n(x) = \frac{1}{2\pi(n+1)}\left(\frac{\sin(\frac{n+1}{2})x}{\sin x/2}\right)^2\ , \quad x \neq 0\ .$$

   (b) Let

$$\sigma_n f(x) = \frac{1}{n+1}\sum_{k=0}^{n} S_k f(x)\ .$$

   Show that for every $x \in [-\pi, \pi]$, $\sigma_n f(x)$ converges uniformly to $f(x)$ for any continuous, $2\pi$-periodic function $f$. Hint: Follow the proof of Theorem 1.5 and use the non-negativity of $F_n$.

   **Solution**. (a) Use $2\sin z/2 \sin(k/2+1)z = \cos kz - \cos(k+1)z$ and $1 - \cos(n+1)z = 2\sin^2\frac{n+1}{2}z$ to get it.

   (b) Note that $\int_{-\pi}^{\pi} F_n(z)\,dz = 1$ as it holds for $D_n$. Proceeding as in the proof of Theorem 1.5,

$$\begin{aligned}
(\sigma_n f)(x) - f(x) &= \int_{-\pi}^{\pi} F_n(z)(f(x+z) - f(x)\,dx \\
&= \frac{1}{2\pi(n+1)}\int_{-\pi}^{\pi}\Phi_\delta(z)\frac{\sin^2(\frac{n+1}{2}z)}{\sin^2 z/2}(f(x+z) - f(x))\,dz \\
&\quad + \frac{1}{2\pi(n+1)}\int_{-\pi}^{\pi}(1 - \Phi_\delta(z))\frac{\sin^2(\frac{n+1}{2}z)}{\sin^2 z/2}(f(x+z) - f(x))\,dz \\
&= I + II\ .
\end{aligned}$$

For the first term, for $\varepsilon > 0$, there is some $\delta$ such that $|f(y) - f(x)| < \varepsilon$, for $y, |y - x| < \delta$. Thus,

$$\begin{aligned} |I| &\leq \left| \int_{-\delta}^{\delta} \Phi_\delta(z) F_n(z)(f(x+z) - f(x))\, dz \right| \\ &\leq \varepsilon \int_{-\delta}^{\delta} F_n(z)\, dz \\ &\leq \varepsilon \int_{-\pi}^{\pi} F_n(z)\, dz \\ &= \varepsilon\ . \end{aligned}$$

The second is easy to handle: For this fixed $\delta$,

$$|II| \leq \frac{1}{2\pi(n+1)} \times \frac{1}{\sin^2 \delta/4} \times 2\|f\|_\infty \to 0\ ,$$

as $n \to \infty$.

**Note** This theorem concerning convergence of the Fourier series in mean was discovered by L Fejèr when he was 19.