THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2024-25
Tutorial 3 solutions
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1. Suppose $f : G \to G'$ with G cyclic, let $g \in G$ be a generator, then any $g' \in G'$ can be written as $g' = f(g^k) = f(g)^k$ for some k, since $f(G) = G'$, we have G' is cyclic. Likewise if G is abelian, for any $c, d \in G'$, there are $a, b \in G$ so that $f(a) = c$ and $f(b) = d$. Then $cd = f(a)f(b) = f(ab) = f(ba) = f(b)f(a) = dc$.

2. Since $H \cap N$ is a subgroup of H, we know $a(H \cap N)a^{-1} = H \cap aNa^{-1} = H \cap N$ for any $a \in H$.

3. Let $G = GL(2, \mathbb{R})$ be the set of 2×2 invertible matrices with coefficients in $\mathbb{R}$, then $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generates a subgroup H isomorphic to $\mathbb{Z}$ since $X^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. Now $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ are conjugate to each other by $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Setting $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, we have $AHA^{-1} \leq H$ is a proper subgroup since $AMA^{-1} = X$ implies M does not have integer coefficients, so cannot possibly lie in H.

4. Let $m \in M$ and $n \in N$, then by normality of N, $mnm^{-1}n^{-1} = (mnm^{-1})n^{-1} \in N$, and by normality of M, $mnm^{-1}n^{-1} = m(nm^{-1}n^{-1}) \in M$. So $mnm^{-1}n^{-1} \in M \cap N = \{e\}$. So $mn = nm$.

5. Let (a_1, a_2) be in the center of $G_1 \times G_2$, then for any $(b_1, b_2) \in G_1 \times G_2$, then $(a_1, a_2)(b_1, b_2) = (b_1, b_2)(a_1, a_2)$ implies that $a_1 b_1 = b_1 a_1$ and $a_2 b_2 = b_2 a_2$. Since b_1, b_2 can be arbitrary elements in G_1, G_2 respectively, this implies $a_1 \in Z_1$ and $a_2 \in Z_2$. The converse is clear.

6. (a) This statement is true in general for S_n. To see why $\tilde{x} = \sigma x \sigma^{-1}$ has the same cycle type as x, think of elements of S_n as a bijective functions on $\{1, ..., n\}$. Writing $x(i) = j$, we have $\tilde{x}(\sigma(i)) = \sigma(x(i)) = \sigma(j)$. So up to relabelling the elements $i \mapsto \sigma(i)$, $\tilde{x}$ and x are the same cycle structure.

   (b) i. By direct counting using combinatorics, there are $24!/4 = 6$ many distinct 4-cycles of the form like (1324).
      ii. There are $24!/3 = 8$ many distinct 3-cycles of form like (241).
      iii. There are $C_2^4/2 = 3$ many distinct $(2, 2)$-cycles of form like $(14)(23)$.
      iv. And there are $C_2^4 = 6$ many distinct 2-cycles of form like (12).
      v. Finally there is one 1-cycle e.

   (c) If $N \leq S_4$ is a normal subgroup, then $\sigma x \sigma^{-1} \in N$ for any $x \in N$ and $\sigma \in S_4$. By the same argument in part (a), if $x, \tilde{x} \in S_4$ are of the same cycle type, then we can find σ so that $\sigma x \sigma^{-1} = \tilde{x}$. Therefore, in order for N to be normal, if we have $x \in N$, it would imply that N contains all elements of the same cycle type as x. At the same time, by Lagrange's theorem, N has order a divisor of $|S_4| = 24$. By examining part (b), we see that the only possible nontrivial proper normal subgroup of S_4 are union of cases (ii), (iii) and (v), or union of cases (iii) and (v). In the first case, it is a subgroup of order 12, hence its index is 2, so it must be normal. In fact, this group is the alternating group A_4 consisting of all even elements of S_4. In the second case, one can check that it is a subgroup, and hence must be a normal subgroup.

(Warning: A priori, we don't know whether taking an arbitrary union of all elements in the same cycle types would form a subgroup. It is necessary to check that it is close under group product.)

7. First of all, we know $x^{[G:N]} \in N$ for any $x \in G$. This is due to Lagrange's theorem: the coset $xN \in G/N$ has order dividing $[G : N] = |G/N|$, therefore $(xN)^{[G:N]} = (x^{[G:H]}H) = H$, i.e. $x^{[G:H]} \in H$. Now the coprime condition implies that there are integers a, b so that $a[G : N] + b \operatorname{ord} N = 1$, therefore

$$x^1 = x^{a\,[G:N]+b\operatorname{ord} N} = (x^{[G:n]})^a (x^{\operatorname{ord} N})^b = (x^{[G:N]})^a \in N.$$

8. Suppose C is a cyclic subgroup that is normal in G, then any subgroup $C' \leq C$ is also a cyclic group. In fact, following from the structures of cyclic groups, C' is *the* unique subgroup of C with that particular order. Therefore for any $g \in G$, $gC'g^{-1} \leq gCg^{-1} = C$ must be equal to C', as $gC'g^{-1}$ has the same cardinality as C'.

9. G' is generated by elements of the form $ABA^{-1}B^{-1}$ for matrices $A, B \in G$, note that $\det(ABA^{-1}B^{-1}) = 1$, and hence every element in G' must lie in $SL(2, \mathbb{R})$.

10. To show that H is normal, note that $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & \frac{-b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix}$, so it suffices to check that $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{a} & \frac{-b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix} \in H$. This is clear since the top left entry is given by $a \cdot 1 \cdot \frac{1}{a} = 1$ and the bottom right entry is given by $c \cdot 1 \cdot \frac{1}{c} = 1$.

    Now consider the commutator subgroup of G, by the same argument as above, if we are given $A, B \in G$, then $ABA^{-1}B^{-1}$ has top left and bottom right entries being 1, hence the commutator subgroup is contained in H. By proposition in lecture 2, we conclude that G/H is abelian.

    To determine the group structure, consider $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ax + b \\ 0 & c \end{pmatrix}$. Therefore in any coset $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} H$, by taking $x = -\frac{b}{a}$, we have a distinguished representative $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Hence $G/H = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} : a \neq 0, c \neq 0 \right\} \cong (\mathbb{R}^\times)^2$.

11. There is relatively simpler proof if you are comfortable with universal properties. Let $F_n = F(a_1, ..., a_n)$ be the free groups on n letters and F_n' be the derived subgroup of F_n, then $F_n^{ab} := F_n/F_n'$ is called the abelianization of F_n. It satisfies the universal property that whenever we are given homomorphism $f : F_n \to G$ to an abelian group G, it factors into

$$\begin{tikzcd} F_n \arrow[r, "f"] \arrow[d, "\pi"'] & G \\ F_n^{ab} \arrow[ur, dashed, "\exists f'"'] & \end{tikzcd}$$

Now let $\mathbb{Z}^{\oplus n}$ be the free abelian group generated by $a_1, ..., a_n$, it satisfies the universal property that given any set function $g : \{a_1, ..., a_n\} \to G$ for abelian group G, there exists a unique homomorphism $f : \mathbb{Z}^{\oplus n} \to G$ extending g. It suffices to prove that F_n^{ab} also satisfies the same universal property. Given $g : \{a_1, ..., a_n\} \to G$ as before, we can first obtain $f : F_n \to G$ by universal property of free group. Since G is abelian, it automatically factors through F_n^{ab}. With the property that $f'(a_i F_n') = f(a_i) := g(a_i)$ for coset $a_i F_n' \in F_n^{ab}$. In particular, since F_n^{ab} and $\mathbb{Z}^{\oplus n}$ are both abelian, we obtain maps $F_n^{ab} \to \mathbb{Z}^{\oplus n}$ and $\mathbb{Z}^{\oplus n} \to F_n^{ab}$ which are inverse to each other. This implies that $F_n^{ab} \cong \mathbb{Z}^{\oplus n}$.

Alternatively, one can prove this statement by defining $F_n \to \mathbb{Z}^{\oplus n}$ by sending generators a_i to $e_i = (0, ..., 1, ..., 0)$ and prove that the kernel of this homomorphism is the same as the commutator subgroup.