THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2024-25
Tutorial 1 Solutions
12th September 2024

- The tutorial solutions are written for reference and proofs will be sketched briefly. You should try to fill in the details as an exercise. The solutions for Homework optional questions can be found in Homework solutions, which would be released after the deadlines. Please send an email to echlam@math.cuhk.edu.hk if you have any further questions.

1. See HW 1 solutions.

2. Suppose $a, b \in G$ are of finite orders, i.e. $a^n = e$ and $b^m = e$ for some m, n, then $(ab)^{mn} = a^{mn}b^{mn} = e$. In $GL(2, \mathbb{R})$ the group of 2×2 invertible matrices, $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ both have order 2, but $AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has infinite order.

   Another example is given by $O(2)$ the set of 2×2 orthogonal matrices, which you should think of as isometry group of the unit circle in $\mathbb{R}^2$. It consists of rotations r_θ by θ and reflections s_ℓ along lines through the origin. For rotations r_θ that is not of rational multiples of π, it has infinite order. One can also see that the composition of two reflections is a rotation, in particular one can find reflections s_1, s_2 so that $s_1 s_2 = r_\theta$ for θ not equal to rational multiple of π.

3. Suppose $(ab)^m = e$, then $(ba)^m = a^{-1}a(ba)^m = a^{-1}(ab)^m a = a^{-1}a = e$, therefore they have the same orders.

4. $\mathbb{Z}_2 \times \mathbb{Z}_3$ is abelian but S_3 is not. Alternatively, you can exhibit an element of the former group with order 6, which does not exist for the latter.

5. Suppose σ is of odd order and is an odd permutation, then σ is given by odd numbers of transpositions, so $\sigma^{ord(\sigma)}$ is also given by an odd numbers of transpositions, but e is obviously even. To put this in a rigorous setting, try to define a homomorphism $S_n \to \mathbb{Z}_2$ capturing the parity of a permutation.

6. Directly check that they define the same bijections $\{1, ..., n\} \to \{1, ..., n\}$. These are known as braid relations.

7. The additive group is divisible while the multiplicative group is not. For example, there does not exist any $g \in (\mathbb{Q}^+, \times)$ such that $g^2 = 2$.

8. An explicitly isomorphism is given by the exponential map $\exp : \mathbb{R} \to \mathbb{R}^+$, clearly $\exp(x + y) = \exp(x)\exp(y)$. It is a bijection function preserving the identity.

9. Let $a \in G$, then there exists b so that $ba = e$, and there also exists c so that $cb = e$. We have $ce = c(ba) = (cb)a = ea = a$. Consider the composition $acbbab$, on one hand $a(cb)(ba)b = ab$. On the other hand, substituting $a = ce$ into the first a gives, $cecbbab = ccbbab = c(cb)(ab)b = cb = e$. So $ab = e$, i.e. any left inverse is also a right

inverse. Now again let $b \in G$ so that $ba = e$, then $ae = a(ba) = (ab)a = ea = a$, so any left identity is also a right identity.

10. Define a product $*$ on $\mathbb{R} - 0$ by $a * b = |a|b$, then it has left identities $1, -1$ and also right inverse: $\forall a$, take $b = 1/|a|$.

11. The set of $n \times n$ invertible matrices with coefficients in $\mathbb{Z}_q$ is in bijection with the set of ordered basis of the underlying vector space $\mathbb{Z}_q^n$. Since $\mathbb{Z}_q^n$ just consists of n-tuples of elements in $\mathbb{Z}_q$, it has cardinality q^n. We can choose any vector except the zero vector as the first column of the matrix, so there are $q^n - 1$ choices. For the second column, we can pick any vector outside the span of the first vector, so there are $q^n - q$ choices. Inductively, for the k^{th} column, we can pick any vector outside the span of the first $k - 1$ vectors, so there are $q^n - q^{k-1}$ choices.

    $GL(2, \mathbb{Z}_2)$ is of order 6 by the above formula. One can see for example $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ that the group is non-abelian, therefore it is isomorphic to S_3. What is the natural 3-element set that it is acting on? It is given by the 3 nontrivial vectors of $\mathbb{Z}_2^2$, i.e. $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.