THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2024-25
Tutorial 1
12th September 2024

- Tutorial exercise would be uploaded to the course webpage on Mondays provided that there is a tutorial class on the coming Thursday. You are not required to hand in the solutions, but you are advised to try the problems before tutorial classes.
- Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

Exercise

1. Optional exercise questions 10, 18, 27, 29 and 32 of Homework 1.
2. Prove that the set of all finite order elements of an **abelian** group G forms a subgroup. Can you find a counterexample in the case when G is non-abelian? (Hint: think about matrices.)
3. Let $a, b \in G$, show that ab and ba have the same order.
4. Are S_3 and $\mathbb{Z}_2 \times \mathbb{Z}_3$ isomorphic? Provide a justification.
5. Let $\sigma \in S_n$ be an element of odd order, show that it must be an even element.
6. For each $1 \leq i < n$, let $s_i \in S_n$ denote the transposition $(i, i+1)$. Show that $s_i s_j = s_j s_i$ if $|i-j| \geq 2$, and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for any i.
7. Prove that the additive group $(\mathbb{Q}, +, 0)$ and the multiplicative group $(\mathbb{Q}^+, \times, 1)$ are **not** isomorphic.
8. Prove that the additive group $(\mathbb{R}, +, 0)$ and the multiplicative group $(\mathbb{R}^+, \times, 1)$ are isomorphic. (Hint: think of a function you have encountered in calculus class.)
9. Suppose that G is a set with product $*$ so that it is associative (i.e. $(G, *)$ is a semigroup), suppose further that it has a left identity and every element has left inverse, i.e.
   (a) There is some $e \in G$ so that $ea = a$ for any $a \in G$.
   (b) For any $a \in G$, there is an element $b \in G$ so that $ba = e$.

   Show that $(G, *)$ is a group.
10. Following Q9, can you find an example of a semigroup G so that it has left identity and right inverse, but G fails to form a group?
11. Let $GL(n, \mathbb{Z}_q)$ denote the set of n by n invertible matrices with coefficients in $\mathbb{Z}_q$, which you can think of as integers modulo a prime number q, prove that it is a group of order $(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$. In particular, $GL(2, \mathbb{Z}_2)$ is a group of order 6, can you figure out its group structure?