**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH 3030 Abstract Algebra 2024-25**
**Homework 8**
**Due Date: 21st November 2024**

## Compulsory Part

1. Prove that if $D$ is an integral domain, then $D[x]$ is an integral domain.

2. Let $D$ be an integral domain and $x$ an indeterminate.

   (a) Describe the units in $D[x]$.

   (b) Find the units in $\mathbb{Z}[x]$.

   (c) Find the units in $\mathbb{Z}_7[x]$.

3. Let $R$ be a commutative ring with unity of prime characteristic $p$. Show that the map $\phi_p : R \to R$ given by $\phi_p(a) = a^p$ is a ring homomorphism (called the **Frobenius homomorphism**).

4. Show that for $p$ a prime, the polynomial $x^p + a$ in $\mathbb{Z}_p[x]$ is reducible for any $a \in \mathbb{Z}_p$.

5. Let $\sigma_m : \mathbb{Z} \to \mathbb{Z}_m$ be the natural reminder homomorphism sending $a$ to the remainder of $a$ when divided by $m$, for $a \in \mathbb{Z}$.

   (a) Show that the induced map $\overline{\sigma}_m : \mathbb{Z}[x] \to \mathbb{Z}_m[x]$ given by

   $$\overline{\sigma}_m(a_0 + a_1x + \cdots + a_nx^n) = \sigma_m(a_0) + \sigma_m(a_1)x + \cdots + \sigma_m(a_n)x^n$$

   is a homomorphism from $\mathbb{Z}[x]$ onto $\mathbb{Z}_m[x]$.

   (b) Show that if $f(x) \in \mathbb{Z}[x]$ and $\overline{\sigma}_m(f(x))$ both have degree $n$ and $\overline{\sigma}_m(f(x))$ does not factor in $\mathbb{Z}_m[x]$ into two polynomials of degree less than $n$, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

   (c) Use part (b) to show that $x^3 + 17x + 36$ is irreducible in $\mathbb{Q}[x]$.

6. Let $\phi : R \to R'$ be a ring homomorphism and let $N$ be an ideal of $R$.

   (a) Show that $\phi(N)$ is an ideal of im $\phi$.

   (b) Given an example to show that $\phi(N)$ need not be an ideal of $R'$.

   (c) Let $N'$ be an ideal of $R'$. Show that $\phi^{-1}(N')$ is an ideal of $R$.

**Optional Part**

1. Let $F$ be a field. An element $\phi$ of $F^F$ is a **polynomial function on** $F$, if there exists $f(x) \in F[x]$ such that $\phi(a) = f(a)$ for all $a \in F$.

   (a) Show that the set $P_F$ of all polynomial functions on $F$ forms a subring of $F^F$.

   (b) Give an example to show that the ring $P_F$ is not necessarily isomorphic to $F[x]$.

2. Give an example to show that, when $F$ is a finite field, $P_F$ and $F[x]$ do not even have the same number of elements.

3. Let $F$ be a field of characteristic zero and let $D$ be the formal polynomial differentiation map, i.e.

$$D(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) := a_1 + 2 \cdot a_2x + \cdots + n \cdot a_nx^{n-1}.$$

   (a) Show that $D : F[x] \to F[x]$ is a group homomorphism from $(F[x], +)$ into itself. Is $D$ a ring homomorphism?

   (b) Find the kernel of $D$.

   (c) Find the image of $F[x]$ under $D$.

4. Let $A$ and $B$ be ideals of a ring $R$. The **product** $AB$ **of** $A$ **and** $B$ is defined by

$$AB = \left\{ \sum_{i=1}^{n} a_ib_i : a_i \in A, b_i \in B, n \in \mathbb{Z}^+ \right\}.$$

   (a) Show that $AB$ is an ideal in $R$.

   (b) Show that $AB \subseteq (A \cap B)$.

5. Let $A$ and $B$ be ideals of a *commutative* ring $R$. The **quotient** $A : B$ **of** $A$ **by** $B$ is defined by

$$A : B = \{r \in R : rb \in A \text{ for all } b \in B\}.$$

   Show that $A : B$ is an ideal of $R$.

6. Let $R$ and $R'$ be rings and let $N$ and $N'$ be ideals of $R$ and $R'$, respectively. Let $\phi$ be a homomorphism of $R$ into $R'$. Show that $\phi$ induces a natural homomorphism $\phi_* : R/N \to R'/N'$ if $\phi(N) \subseteq N'$.