THE CHINESE UNIVERSITY OF HONG KONG Department of Mathematics MATH 3030 Abstract Algebra 2024-25 Homework 7 Due Date: 7th November 2024

Compulsory Part

- 1. Let G be a finite group, and suppose that there exist representatives g_1, \ldots, g_r of the r distinct conjugacy classes in G such that $g_i g_j = g_j g_i$ for all i, j. Show that G is abelian.
- 2. Let G be a finite group and let primes p and $q \neq p$ divide |G|. Prove that if G has precisely one proper Sylow p-subgroup, then it must be a normal subgroup, and hence G is not simple.
- 3. Let G be a finite group and let p be a prime dividing |G|. Let P be a Sylow p-subgroup of G.
 - (a) Show that P is the only Sylow p-subgroup of $N_G(N_G(P))$.
 - (b) Using part (a) and applying Sylow Theorems, show that $N_G(N_G(P)) = N_G(P)$.
- 4. Show that there are no simple groups of order $p^r m$, where p is a prime, r is a positive integer, and 1 < m < p.
- 5. Let G be a group of order 6. Suppose G is not abelian.
 - (a) Show that G has three subgroups of order 2.
 - (b) Show that there is a homomorphism φ : G → S₃ with |ker(φ)| ≤ 2. [*Hint:* Consider the action of G on the set of left cosets of a subgroup of order 2 in G (as in HW6, Optional Q.5).]
 - (c) Show that $G \simeq S_3$.
- 6. (a) Let G be a finite group, and H, K < G. Show that

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}.$$

(Note that HK may not be a subgroup of G, so the above is just an equality between orders of sets.)

- (b) Suppose that G is a finite group of order 48.
 - i. Applying Sylow Theorems, show that the number n_2 of Sylow 2-subgroups in G is either 1 or 3.
 - ii. Suppose that $n_2 = 3$ and let H, K be two distinct Syloew 2-subgroups in G. Show that $|H \cap K| = 8$ by applying part (a). From this and considering the normalizer $N_G(H \cap K)$, deduce that $H \cap K$ is normal in G, thereby showing that G cannot be simple.

Optional Part

- 1. Let G be a finite group of odd order. Suppose that $g \in G$ and g^{-1} lie in the same conjugacy class. Show that g = e.
- 2. Show that every group of order 30 contains a subgroup of order 15.
- 3. Prove that no group of order 160 is simple.
- 4. How many elements of order 7 are there in a simple group of order 168?
- 5. Let p, q be prime numbers. Show that a group of order p^2q is solvable.
- 6. Let p < q < r be prime numbers. Show that a group of order pqr is not simple.