THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2024-25
Homework 10
Due Date: 5th December 2024

Compulsory Part

1. Prove that if p is an irreducible in a UFD, then p is a prime.

2. Let D be a UFD. Show that a non-constant divisor of a primitive polynomial in $D[x]$ is again a primitive polynomial.

3. Let R be any ring. The **ascending chain condition (ACC) for ideals** holds in R if every strictly increasing sequence $N_1 \subset N_2 \subset N_3 \subset \cdots$ of ideals in R is of finite length. The **maximum condition (MC) for ideals** holds in R if every non-empty set S of ideals in R contains an ideal not properly contained in any other ideal of the set S. The **finite basis condition (FBC) for ideals** holds in R if for each ideal N in R, there is a finite set $B_N = \{b_1, \cdots, b_n\} \subseteq N$ such that N is the intersection of all ideals of R containing B_N. The B_N is a **finite generating set for** N.
   Show that for every ring R, the conditions ACC, MC, and FBC are equivalent.

4. Prove or disprove the following statement: If ν is a Euclidean norm on Euclidean domain D, then $\{a \in D : \nu(a) > \nu(1)\} \cup \{0\}$ is an ideal of D.

5. Let $\langle \alpha \rangle$ be a non-zero principal ideal in $\mathbb{Z}[i]$.
   (a) Show that $\mathbb{Z}[i]/\langle \alpha \rangle$ is a finite ring.
   (b) Show that if π is an irreducible of $\mathbb{Z}[i]$, then $\mathbb{Z}[i]/\langle \pi \rangle$ is a field.
   (c) Referring to part b, find the order and characteristic of each of the following fields.
      i. $\mathbb{Z}[i]/\langle 3 \rangle$
      ii. $\mathbb{Z}[i]/\langle 1+i \rangle$
      iii. $\mathbb{Z}[i]/\langle 1+2i \rangle$

6. Let $n \in \mathbb{Z}^+$ be square free, that is , not divisible by the square of any prime integer. Let $\mathbb{Z}[\sqrt{-n}] = \{a + ib\sqrt{n} | a, b \in \mathbb{Z}\}$.
   (a) Show that the norm N, defined by $N(\alpha) = a^2 + nb^2$ for $\alpha = a + ib\sqrt{n}$, is a multiplicative norm on $\mathbb{Z}[\sqrt{-n}]$.
   (b) Show that $N(\alpha) = 1$ for $\alpha \in \mathbb{Z}[\sqrt{-n}]$ if and only if α is a unit of $\mathbb{Z}[\sqrt{-n}]$.
   (c) Show that every non-zero $\alpha \in \mathbb{Z}[\sqrt{-n}]$ that is not a unit has a factorization into irreducibles in $\mathbb{Z}[\sqrt{-n}]$.

Optional Part

1. Let R be any ring. The **descending chain condition (DCC) for ideals** holds in R if every strictly decreasing sequence $N_1 \supset N_2 \supset N_3 \supset \cdots$ of ideals in R is of finite length. The **minimum condition (mC) for ideals** holds in R if given any set S of ideals of R, there is an ideal of S that does not properly contain any other ideal in the set S.
   Show that for every ring, the conditions DCC and mC are equivalent.

2. Give an example of a ring in which ACC holds but DCC does not hold.

3. Show that every field is a Euclidean domain.

4. Let ν be a Euclidean norm on a Euclidean domain D.

   a. Show that if $s \in \mathbb{Z}$ such that $s + \nu(1) > 0$, then $\eta : D^* \to \mathbb{Z}$ defined by $\eta(a) = \nu(a) + s$ for non-zero $a \in D$ is a Euclidean norm on D. As usual, D^* is the set of non-zero elements of D.

   b. Show that for $t \in \mathbb{Z}^+$, $\lambda : D^* \to \mathbb{Z}$ given by $\lambda(a) = t \cdot \nu(a)$ for non-zero $a \in D$ is a Euclidean norm on D.

   c. Show that there exists a Euclidean norm μ on D such that $\mu(1) = 1$ and $\mu(a) > 100$ for all non-zero non-units $a \in D$.

5. Let D be a UFD. Show that all common multiples, in the obvious sense, of both a and b form an ideal of D.

6. Let D be a UFD. An element c in D is a **least common multiple** (abbreviated lcm) of two elements a and b in D if $a|c, b|c$ and if c divides every element of D that is divisible by both a and b. Show that every two non-zero elements a and b of a Euclidean domain D have an lcm in D.