THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2024-25
Homework 1
Due Date: 12th September 2024

Many of these exercises are adopted from the textbook or reference books. You are suggested to work out more from these or relevant books.

Compulsory Part

1. A nontrivial abelian group A (written multiplicatively) is called **divisible** if for each element $a \in A$ and each nonzero integer k there is an element $x \in A$ such that $x^k = a$, i.e. each element has a k^{th} root in A.

   (a) Prove that the additive group of rational numbers, $\mathbb{Q}$, is divisible.

   (b) Prove that no finite abelian group is divisible.

2. Let G be a group of order pq, where p and q are primes. Show that every proper subgroup of G is cyclic.

3. Let $H_1 \leq H_2 \leq H_3 \ldots$ be an ascending chain of subgroups of a group G. Prove that the union $\cup_{i=1}^{\infty} H_i$ is a subgroup of G.

4. Let $H \leq K \leq G$. Show that $[G : H] = [G : K][K : H]$. (*Warning:* G, H and K may *not* be finite.)

5. Show that if H is a subgroup of index 2 in a group G, then $aH = Ha$ (as subsets in G) for all $a \in G$. (*Warning:* Again, G may *not* be finite.)

6. Show that any group homomorphism $\phi : G \to G'$, where $|G|$ is a prime number, must either be the trivial homomorphism or an injective map.

Optional Part

1. Recall that an element a of a group G with identity element e has **order** $r > 0$ if $a^r = e$ and no smaller positive power of a is the identity. Show that if G is a finite group with identity e and with an even number of elements, then there exists an order 2 element in G, i.e. there exists $a \neq e$ in G such that $a^2 = e$.

2. Using the Theorem of Lagrange, show that if n is odd, then an abelian group of order $2n$ contains precisely one element of order 2.

3. Show that every group G with identity e and such that $x^2 = e$ for all $x \in G$ is abelian.

4. Let p be a prime and $\mathbb{F}_p$ be the finite field with p elements. Compute the orders of the groups $GL_n(\mathbb{F}_p)$ and $SL_n(\mathbb{F}_p)$.

5. Prove that a cyclic group with *only one* generator can have at most 2 elements.

6. Show that a group with no proper nontrivial subgroups is cyclic.

7. Show that a group which has only a finite number of subgroups must be a finite group.

8. Let G be a group and suppose that an element $a \in G$ generates a cyclic subgroup of order 2 and is the *unique* such element. Show that $ax = xa$ for all $x \in G$. [*Hint:* Consider $(xax^{-1})^2$.]

9. Let n be an integer greater than or equal to 3. Show that the only element σ of S_n satisfying $\sigma g = g\sigma$ for all $g \in S_n$ is $\sigma = \iota$, the identity permutation. [*Hint:* First show that S_n is a nonabelian group for $n \geq 3$.]

10. Prove the following statements about S_n for $n \geq 3$:

    (a) Every permutation in S_n can be written as a product of at most $n-1$ transpositions.

    (b) Every permutation in S_n that is not a cycle can be written as a product of at most $n-2$ transpositions.

    (c) Every odd permutation in S_n can be written as a product of $2n+3$ transpositions, and every even permutation as a product of $2n+8$ transpositions.

11. Show that if $\sigma \in S_n$ is a cycle of odd length, then σ^2 is a cycle.

12. If n is odd and $n \geq 3$, show that the identity is the only element of D_n which commutes with all elements of D_n.

13. Consider the group S_8.

    (a) What is the order of the cycle $(1, 4, 5, 7)$?

    (b) State a theorem suggested by part (a).

    (c) What is the order of $\sigma = (4, 5)(2, 3, 7)$? of $\tau = (1, 4)(3, 5, 7, 8)$?

    (d) Find the order of each of the permutations given in Exercise 13 (a) through (c) (see below) by looking at its decomposition into a product of disjoint cycles.

(e) State a theorem suggested by parts (c) and (d). [*Hint:* The important words you are looking for are *least common multiple*.]

14. Express the permutation of $\{1,2,3,4,5,6,7,8\}$ as a product of disjoint cycles, and then as a product of transpositions:

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix}$

15. Find the maximum possible order for an element of S_6.

16. Find the maximum possible order for an element of S_{10}.

17. Complete the following with a condition involving n and r so that the resulting statement is a theorem:

If σ is a cycle of length n, then σ^r is also a cycle if and only if...

18. Show that S_n is generated by $\{(1,2),(1,2,3,\dots,n)\}$.

[*Hint:* Show that as r varies, $(1,2,3,\dots,n)^r(1,2)(1,2,3,\dots,n)^{n-r}$ gives all the transpositions $(1,2),(2,3),(3,4),\cdots,(n-1,n),(n,1)$. Then show that any transposition is a product of some of these transpositions and use Corollary 9.12.]

19. Prove that $\mathbb{Q}\times\mathbb{Q}$ is not cyclic.

20. Exhibit a proper subgroup of $\mathbb{Q}$ which is not cyclic.

21. Show that if a group G with identity e has finite order n, then $a^n = e$ for all $a \in G$.

22. Let H and K be subgroups of a group G. Define a relation $\sim$ on G by $a \sim b$ if and only if $a = hbk$ for some $h \in H$ and some $k \in K$.

(a) Prove that $\sim$ is an equivalence relation on G.

(b) Describe the elements in the equivalence class containing $a \in G$. (These equivalence classes are called **double cosets**.)

23. Let H and K be subgroups of finite index in a group G, and suppose that $[G:H] = m$ and $[G:K] = n$. Prove that $\text{lcm}(m,n) \le [G : H\cap K] \le mn$. Hence deduce that if m and n are relatively prime, then $[G : H\cap K] = [G:H][G:K]$.

24. Let $\phi : G \to G'$ be a homomorphism with kernel H and let $a \in G$. Prove the set equality $\{x \in G : \phi(x) = \phi(a)\} = Ha$.

25. Show that a nontrivial group which has no proper nontrivial subgroups must be finite and of prime order.

26. If A and B are groups, then their Cartesian product $A \times B$ is a group (called the **direct product** of A and B) using the componentwise defined operation. Is any subgroup of $A \times B$ of the form $C \times D$ where $C < A$ and $D < B$? Justify your assertion.

27. Prove, carefully and rigorously, that a finite cyclic group of order n has exactly one subgroup of each order d dividing n.

28. The **sign of an even permutation** is $+1$ and the **sign of an odd permutation** is -1. Observe that the map $\text{sgn}_n : S_n \to \{1, -1\}$ defined by

$$\text{sgn}_n(\sigma) = \text{sign of } \sigma$$

is a homomorphism of S_n onto the multiplicative group $\{1, -1\}$. What is the kernel?

29. Let $\phi : G_1 \to G_2$ be a group homomorphism. Show that ϕ induces an order preserving one-to-one correspondence between the set of all subgroups of G_1 that contain $\ker \phi$ and the set of all subgroups of G_2 that are contained in $\text{im}\, \phi$.

30. Let G be a group, let $h, k \in G$ and let $\phi : \mathbb{Z} \times \mathbb{Z} \to G$ be defined by $\phi(m, n) = h^m k^n$. Give a necessary and sufficient condition, involving h and k, for ϕ to be a homomorphism. Prove your assertion.

31. Find a necessary and sufficient condition on G such that the map ϕ described in the preceding exercise is a homomorphism for *all* choices of $h, k \in G$.

32. Let G be a group, h be an element of G, and n be a positive integer. Let $\phi : \mathbb{Z}_n \to G$ be defined by $\phi(i) = h^i$ for $0 \leq i < n$. Give a necessary and sufficient condition (in terms of h and n) for ϕ to be a homomorphism. Prove your assertion.