THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230B/C - Complex Variables with Applications - 2024/25 Term 2

Homework 6
Due: Friday 28th February 2025, 23:59pm.

Note. For those of you who may not have the textbook, I have manually typed the questions below. However, make sure you always double check for typos if possible, and in which case please inform me by sending an email to bwang@math.cuhk.edu.hk; I would correct them immediately.

1. (P. 147, Q5) Show that

$$\int_{-1}^{1} z^i \, dz = \frac{1+e^{-\pi}}{2}(1-i),$$

where the integrand denotes the principal branch

$$z^i = \exp(i \operatorname{Log} z) \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis.

Suggestion: Use an antiderivative of the branch

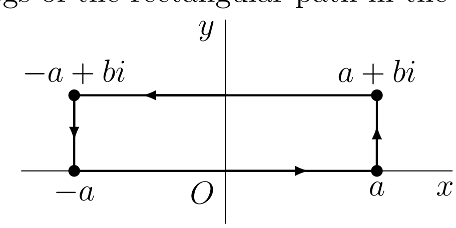
$$z^i = \exp(i \log z) \quad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the same power function.

2. (P. 159, Q4) Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2} \cos 2bx \, dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \quad (b > 0).$$

(a) Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path in the following figure [1]



FIGURE

can be written

$$2\int_0^a e^{-x^2} \, dx - 2e^{b^2} \int_0^a e^{-x^2} \cos 2bx \, dx$$

[1]Thanks to Samuel Ho for helping with the nice plotting.

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}\,dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}\,dy.$$

Thus, with the aid of the Cauchy-Goursat theorem, show that

$$\int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}\,dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy.$$

(b) By accepting the fact that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\left|\int_0^b e^{y^2}\sin 2ay\,dy\right| \le \int_0^b e^{y^2}\,dy,$$

obtain the desired integration formula by letting a tend to infinity in the equation at the end of part (a).

3. (P. 159, Q6) Let C denote the positively oriented boundary of the half disk $0 \le r \le 1, 0 \le \theta \le \pi$, and let $f(z)$ be a continuous function defined on that half disk by writing $f(0) = 0$ and using the branch

$$f(z) = \sqrt{r}e^{i\theta/2} \quad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{1/2}$. Show that

$$\int_C f(z)\,dz = 0$$

by evaluating separately the integrals of $f(z)$ over the semicircle and the two radii which make up C. Why does the Cauchy-Goursat theorem not apply here?