MATH 2068 Honours Mathematical Analysis II
2024-25 Term 2
Suggested Solution to Homework 9

9.3-1 Test the following series for convergence and for absolute convergence:

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n+2}$, (d) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$.

Solution. (c) Since $\lim \left|\dfrac{(-1)^{n+1}n}{n+2}\right| = \lim \dfrac{n}{n+2} = 1 \neq 0$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n+2}$ is not convergent and not absolutely convergent, by the nth Term Test.

(d) Let $f(x) = \dfrac{\ln x}{x}$. Then $f'(x) = \dfrac{1-\ln x}{x^2} < 0$ when $x \geq 3$. So $(n^{-1}\ln n)_{n=3}^{\infty}$ is a decreasing sequence of strictly positive numbers. By the Alternating Test, the series $\displaystyle\sum_{n=3}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$ is convergent, and so is $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$.

Note that

$$r := \lim_{n\to\infty}\left|\frac{n^{-1}}{(-1)^{n+1}n^{-1}\ln n}\right| = \lim_{n\to\infty}\frac{1}{\ln n} = 0,$$

and the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent (so not absolutely convergent). By the Limit Comparison Test II (Theorem 9.2.1 of the textbook), the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$ is not absolutely convergent.

□

9.4-6 Determine the radius of convergence of the series $\sum a_n x^n$, where a_n is given by:

(c) $n^n/n!$, (d) $(\ln n)^{-1}$, $n \geq 2$.

Solution. (c) The radius of convergence of the series $\sum a_n x^n$ is

$$R := \lim\left|\frac{a_n}{a_{n+1}}\right| = \lim\frac{1}{(1+1/n)^n} = \frac{1}{e}.$$

(d) The radius of convergence of the series $\sum a_n x^n$ is

$$R := \lim\left|\frac{a_n}{a_{n+1}}\right| = \lim\frac{\ln(n+1)}{\ln n} = \lim\left(1 + \frac{\ln(1+1/n)}{\ln n}\right) = 1.$$

□