MATH 2068 Honours Mathematical Analysis II
2024-25 Term 2
Suggested Solution to Homework 8

8.2-1 Show that the sequence $(x^n/(1+x^n))$ does not converge uniformly on $[0,2]$ by showing that the limit function is not continuous on $[0,2]$.

Solution. It is easy to see that

$$\lim\left(x^n/(1+x^n)\right) = f(x) := \begin{cases} 0 & \text{if } 0 \le x < 1; \\ 1/2 & \text{if } x = 1; \\ 1 & \text{if } 1 < x \le 2. \end{cases}$$

If the sequence $(x^n/(1+x^n))$ of continuous functions converges uniformly on $[0,2]$, then the limit function f is also continuous on $[0,2]$, by Theorem 8.2.2 of the textbook. However, f is clearly discontinuous at 1. Therefore the sequence $(x^n/(1+x^n))$ does not converge uniformly on $[0,2]$. $\square$

8.2-9 Let $f_n(x) := x^n/n$ for $x \in [0,1]$. Show that the sequence (f_n) of differentiable functions converges uniformly to a differentiable function f on $[0,1]$, and that the sequence (f_n') converges on $[0,1]$ to a function g, but that $g(1) \neq f'(1)$.

Solution. For each $n \in \mathbb{N}$, we have $|f_n(x) - 0| = |x|^n/n \le 1/n$ for all $x \in [0,1]$, and so

$$\|f_n - 0\|_{[0,1]} \le 1/n \to 0 \quad \text{as } n \to \infty.$$

Hence (f_n) converges uniformly to the zero function $f \equiv 0$ (clearly differentiable) on $[0,1]$.

For the sequence (f_n'), we have

$$\lim_{n\to\infty}\left(f_n'(x)\right) = \lim_{n\to\infty} x^{n-1} = g(x) := \begin{cases} 0 & \text{if } 0 \le x < 1; \\ 1 & \text{if } x = 1. \end{cases}$$

However, $g(1) = 1$ while $f'(1) = 0$. $\square$

9.2-3 Discuss the convergence or the divergence of the series with nth term (for sufficiently large n) given by:

(e) $(n \ln n)^{-1}$

(f) $(n(\ln n)(\ln \ln n)^2)^{-1}$

Solution. (e) The function $f(x) := (x \ln x)^{-1}$ is positive and decreasing on $[2,\infty)$. Moreover,

$$\int_2^\infty f(t)\,dt = \lim_{b\to\infty}\int_2^b f(t)\,dt = \lim_{b\to\infty}\int_2^b \frac{1}{t\ln t}\,dt = \lim_{b\to\infty}\left[\ln(\ln t)\right]_2^b = \infty.$$

By the Integral Test (Theorem 9.2.6 of the textbook), the series $\sum f(n) = \sum (n \ln n)^{-1}$ is divergent.

(f) The function $f(x) := (x(\ln x)(\ln\ln x)^2)^{-1}$ is positive and decreasing on $[4, \infty)$. Moreover,

$$\int_4^\infty f(t)\,dt = \lim_{b\to\infty} \int_4^b f(t)\,dt = \lim_{b\to\infty} \int_4^b \frac{1}{t(\ln t)(\ln\ln t)^2}\,dt = \lim_{b\to\infty} \Big[\frac{-1}{\ln\ln t}\Big]_4^b = \frac{1}{\ln\ln 4}.$$

By the Integral Test (Theorem 9.2.6 of the textbook), the series $\sum f(n) = \sum (n(\ln n)(\ln\ln n)^2)^{-1}$ is convergent.

$\square$