MATH 2068 Honours Mathematical Analysis II
2024-25 Term 2
Suggested Solution to Homework 7

8.1-4 Evaluate $\lim(x^n/(1+x^n))$ for $x \in \mathbb{R}$, $x \geq 0$.

Solution. If $0 \leq x < 1$, then $\lim(x^n/(1+x^n)) = 0/(1+0) = 0$.

If $x = 1$, then $\lim(x^n/(1+x^n)) = 1/(1+1) = 1/2$.

If $x > 1$, then $\lim(x^n/(1+x^n)) = \lim(1/(x^{-n}+1)) = 1/(0+1) = 1$. $\square$

8.1-14 Show that if $0 < b < 1$, then the convergence of the sequence in Exercise 4 is uniform on the interval $[0, b]$, but is not uniform on the interval $[0, 1]$.

Solution. Let (f_n) be the sequence of functions considered in Exercise 4, and let f be its limit. Since $0 \leq f_n(x) = \frac{x^n}{1+x^n} \leq \frac{b^n}{1+0} = b^n$ for any $x \in [0, b]$, we have

$$\|f_n - 0\|_{[0,b]} \leq b^n \quad \text{for all } n \in \mathbb{N}.$$

As $0 < b < 1$, we have $\lim(b^n) = 0$ and so $\lim\|f_n - 0\|_{[0,b]} = 0$. Therefore (f_n) converges uniformly to $f \equiv 0$ on $[0, b]$.

On the other hand, for all $n \in \mathbb{N}$,

$$\|f_n - f\|_{[0,1]} \geq |f_n(2^{-1/n}) - f(2^{-1/n})| = \left|\frac{1/2}{1+1/2} - 0\right| = \frac{1}{3}.$$

So $\|f_n - f\|_{[0,1]} \not\to 0$ as $n \to \infty$. Therefore (f_n) does not converge uniformly to f on $[0, 1]$. $\square$

8.1-20 Show that if $a > 0$, then the sequence $(n^2x^2e^{-nx})$ converges uniformly on the interval $[a, \infty)$, but that it does not converge uniformly on the interval $[0, \infty)$.

Solution. If $0 < a \leq x$, then $e^{nx} \geq \frac{1}{6}(nx)^3 \geq \frac{n^3a}{6}x^2$, and hence

$$0 \leq f_n(x) \coloneqq n^2x^2e^{-nx} \leq \frac{6}{an}.$$

Thus $\lim\|f_n - 0\|_{[a,\infty)} = 0$ and (f_n) converges uniformly to the zero function 0 on $[a, \infty)$.

In particular, if (f_n) converges uniformly to f on $[0, \infty)$, then $f(x) = 0$ for $x > 0$. However, for all $n \in \mathbb{N}$,

$$\|f_n - f\|_{[0,\infty)} \geq |f_n(1/n) - f(1/n)| = e^{-1} > 0.$$

Therefore, (f_n) does not converge uniformly on $[0, \infty)$. $\square$