MATH 2068 Honours Mathematical Analysis II
2024-25 Term 2
Suggested Solution to Homework 6

7.3-17 Let $J := [\alpha, \beta]$, let $\varphi : J \to \mathbb{R}$ have a continuous derivative on J, and let $f : I \to \mathbb{R}$ be continuous on an interval I containing $\varphi(J)$.

Use the following argument to prove the Substitution Theorem 7.3.8.

Define $F(u) := \int_{\varphi(\alpha)}^{u} f(x)\,dx$ for $u \in I$, and $H(t) := F(\varphi(t))$ for $t \in J$. Show that $H'(t) = f(\varphi(t))\varphi'(t)$ for $t \in J$ and that

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = F(\varphi(\beta)) = H(\beta) = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)\,dt.$$

Solution. By extending f, we may assume that I is a closed interval $[a, b]$ and $\varphi(J) \subseteq (a, b)$. (We do this because the Fundamental Theorem of Calculus in the note is slightly different from the one in the textbook.)

Since f is continuous on $[a, b]$, the Fundamental Theorem of Calculus (Theorem 2.25(ii)) implies that $F(u)$ is differentiable on (a, b) and $F' = f$ on (a, b). By Chain Rule (Proposition 1.6), $H = F \circ \varphi$ is differentiable on J and

$$H'(t) = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t) \quad \text{for } t \in J.$$

Note that H' is continuous on J, and so $H' \in \mathcal{R}[\alpha, \beta]$. Hence, by the Fundamental Theorem of Calculus (Theorem 2.25(i)) again,

$$\int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)\,dt = \int_{\alpha}^{\beta} H'(t)\,dt = H(\beta) - H(\alpha).$$

Since $H(\alpha) = \int_{\varphi(\alpha)}^{\varphi(\alpha)} f(x)\,dx = 0$, we have

$$\int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx.$$

$\square$

7.3.21 Let $f, g \in \mathcal{R}[a, b]$.

(a) If $t \in \mathbb{R}$, show that $\int_a^b (tf \pm g)^2 \geq 0$.

(b) Use (a) to show that $2\left|\int_a^b fg\right| \leq t\int_a^b f^2 + (1/t)\int_a^b g^2 \quad$ for $t > 0$.

(c) If $\int_a^b f^2 = 0$, show that $\int_a^b fg = 0$.

(d) Now prove that $\left|\int_a^b fg\right|^2 \leq \left(\int_a^b |fg|\right)^2 \leq \left(\int_a^b f^2\right) \cdot \left(\int_a^b g^2\right)$. This inequality is called the **Cauchy-Bunyakovsky-Schwarx Inequality** (or simply the **Schwarz Inequality**).

Solution. (a) For $t \in \mathbb{R}$, we have $(tf \pm g)^2 \geq 0$, and so $\int_a^b (tf \pm g)^2 \geq 0$ by Proposition 2.14(i).

(b) Since $f, g \in \mathcal{R}[a, b]$, we have $f^2, fg, g^2 \in \mathcal{R}[a, b]$ by Proposition 2.19. The linearity of integral (Proposition 2.9) then implies that

$$0 \leq \int_a^b (tf \pm g)^2 = t^2 \int_a^b f \pm 2t \int_a^b fg + \int_a^b g^2.$$

Thus, for $t > 0$, we have

$$-t \int_a^b f^2 - (1/t) \int_a^b g^2 \leq 2 \int_a^b fg \leq t \int_a^b f^2 + (1/t) \int_a^b g^2,$$

that is $2 \left| \int_a^b fg \right| \leq t \int_a^b f^2 + (1/t) \int_a^b g^2$.

(c) By (b), we have $\left| \int_a^b fg \right| \leq (1/2t) \int_a^b g^2$ for $t > 0$. Letting $t \to +\infty$ yields $\left| \int_a^b fg \right| \leq 0$, that is $\int_a^b fg = 0$.

(d) If $\int_a^b f^2 = 0$ or $\int_a^b g^2 = 0$, we have $\int_a^b fg = 0$ by (c). If $\int_a^b f^2 > 0$ and $\int_a^b g^2 > 0$, taking $t = \sqrt{\int_a^b g^2} / \sqrt{\int_a^b f^2} > 0$ in (b) yields

$$2 \left| \int_a^b fg \right| \leq 2 \sqrt{\int_a^b f^2} \sqrt{\int_a^b g^2},$$

that is $\left| \int_a^b fg \right| \leq \left(\int_a^b f^2 \right) \cdot \left(\int_a^b g^2 \right)$. The desired inequalities follow if we replace f, g by $|f|, |g|$ respectively.

$\square$