MATH 2068 Honours Mathematical Analysis II
2024-25 Term 2
Suggested Solution to Homework 5

7.2-10 If f and g are continuous on $[a,b]$ and if $\int_a^b f = \int_a^b g$, prove that there exists $c \in [a,b]$ such that $f(c) = g(c)$.

Solution. Suppose $f(x) \neq g(x)$ for any $x \in [a,b]$. Then the Intermediate Value Theorem implies that either $f - g > 0$ or $g - f > 0$ on $[a,b]$. Together with $\displaystyle\int_a^b (f-g) = \int_a^b f - \int_a^b g = 0$, Exercise 7.2-8 (see HW4) implies that $f - g = 0$ on $[a,b]$, which contradicts the assumption at the beginning. □

7.2-12 Show that $g(x) := \sin(1/x)$ for $x \in (0,1]$ and $g(0) := 0$ belongs to $\mathcal{R}[0,1]$.

Solution. Clearly $|g(x)| \leq 1$ for all $x \in [0,1]$.

Let $\varepsilon > 0$. Choose $c \in (0,1)$ such that $c < \varepsilon/4$. On $[c,1]$, $g(x) = \sin(1/x)$ is continuous, and hence $g \in \mathcal{R}[c,1]$ by Proposition 2.13. By Theorem 2.10, there is a partition $P : c = x_1 < \cdots < x_n = 1$ on $[c,1]$ such that

$$0 \leq U(g,P) - L(g,P) = \sum_{i=1}^{n} \omega_i(g,P)\Delta x_i < \varepsilon/2,$$

where $\omega_i(g,P) := \sup\{|g(x) - g(x')| : x, x' \in [x_{i-1}, x_i]\}$. Now $P' : 0 =: x_0 < x_1 = c < x_2 < \cdots < x_n = 1$ is a partition on $[0,1]$ that satisfies

$$\begin{aligned} 0 \leq U(g,P') - L(g,P') &= \sum_{i=1}^{n} \omega_i(g,P')\Delta x_i \\ &= \sup\{|g(x) - g(x')| : x, x' \in [0,c]\}(c - 0) + \sum_{i=2}^{n} \omega_i(g,P)\Delta x_i \\ &< 2(\varepsilon/4) + \varepsilon/2 = \varepsilon. \end{aligned}$$

By Theorem 2.10 again, $g \in \mathcal{R}[0,1]$. □

7.2-18 Let f be continuous on $[a,b]$, let $f(x) \geq 0$ for $x \in [a,b]$, and let $M_n := (\int_a^b f^n)^{1/n}$. Show that $\lim(M_n) = \sup\{f(x) : x \in [a,b]\}$.

Solution. Denote $\|f\|_\infty := \sup\{|f(x)| : x \in [a,b]\} = \sup\{f(x) : x \in [a,b]\}$. Without loss of generality, we may assume that $\|f\|_\infty > 0$.

Let $0 < \varepsilon < \|f\|_\infty$. By definition of supremum, there is $x_0 \in [a,b]$ such that $f(x_0) > \|f\|_\infty - \varepsilon/2$. Since f is continuous at x_0, there is a subinterval $[c,d] \subseteq [a,b]$ such that

$$f(x) > f(x_0) - \varepsilon/2 \geq \|f\|_\infty - \varepsilon > 0 \quad \text{for any } x \in [c,d].$$

So, for any $n \in \mathbb{N}$,

$$(d-c)(\|f\|_\infty - \varepsilon)^n = \int_c^d (\|f\|_\infty - \varepsilon)^n \leq \int_c^d f^n \leq \int_a^b f^n \leq (b-a)\|f\|_\infty^n,$$

and thus,

$$(d-c)^{1/n}(\|f\|_\infty - \varepsilon) \le M_n = \left(\int_a^b f^n\right)^{1/n} \le (b-a)^{1/n}\|f\|_\infty.$$

As $\lim_n \alpha^{1/n} = 1$ for any $\alpha > 0$, passing $n \to \infty$ yields

$$\|f\|_\infty - \varepsilon \le \liminf_n M_n \le \limsup_n M_n \le \|f\|_\infty.$$

Since $\varepsilon > 0$ can be arbitrarily small, we have $\liminf_n (M_n) = \limsup_n (M_n) = \|f\|_\infty$, that is

$$\lim(M_n) = \|f\|_\infty = \sup\{f(x) : x \in [a,b]\}.$$

$\square$