MATH 2068 Honours Mathematical Analysis II
2024-25 Term 2
Suggested Solution to Homework 4

7.1-6 (a) Let $f(x) := 2$ if $0 \le x < 1$ and $f(x) := 1$ if $1 \le x \le 2$. Show that $f \in \mathcal{R}[0,2]$ and evaluate its integral.

(b) Let $h(x) := 2$ if $0 \le x < 1$, $h(1) := 3$ and $h(x) := 1$ if $1 < x \le 2$. Show that $h \in \mathcal{R}[0,2]$ and evaluate its integral.

Solution. Fix $c \in \mathbb{R}$ and define $g : [0,2] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 2 & \text{if } 0 \le x < 1; \\ c & \text{if } x = 1; \\ 1 & \text{if } 1 < x \le 2. \end{cases}$$

We will show that, regardless of the value of c, we always have $g \in \mathcal{R}[0,2]$ and $\int_0^2 g = 3$.

For $\varepsilon \in (0,1)$, let $\mathcal{P}_\varepsilon$ be the partition

$$0 < 1-\varepsilon < 1+\varepsilon < 2.$$

Then

$$\overline{\int_0^2} f \le U(f,\mathcal{P}_\varepsilon) = 2(1-\varepsilon) + \max\{2,1,c\}(2\varepsilon) + 1(1-\varepsilon) = 3 + (2\max\{2,c\} - 3)\varepsilon,$$

and

$$\underline{\int_0^2} f \ge L(f,\mathcal{P}_\varepsilon) = 2(1-\varepsilon) + \min\{2,1,c\}(2\varepsilon) + 1(1-\varepsilon) = 3 + (2\min\{1,c\} - 3)\varepsilon.$$

Since $\underline{\int_0^2} f \le \overline{\int_0^2} f$, letting $\varepsilon \to 0^+$ yields $\underline{\int_0^2} f = \overline{\int_0^2} f = 3$. Therefore $f \in \mathcal{R}[0,2]$ and $\int_0^2 f = 3$.

□

7.2-2 Consider the function h defined by $h(x) := x+1$ for $x \in [0,1]$ rational, and $h(x) := 0$ for $x \in [0,1]$ irrational. Show that h is not Riemann integrable.

Solution. By the density of rational and irrational numbers,

$$\sup_{a \le x \le b} h(x) = b+1, \qquad \text{and} \qquad \inf_{a \le x \le b} h(x) = 0.$$

For any partition $P : 0 = x_0 < x_1 < \cdots < x_n = 1$, we have

$$U(h,P) = \sum_{i=1}^{n} (x_i + 1)(x_i - x_{i-1}) \ge \sum_{i=1}^{n} (x_i - x_{i-1}) = 1,$$

and

$$L(h,P) = \sum_{i=1}^{n} (0)(x_i - x_{i-1}) = 0.$$

Thus

$$\underline{\int_0^1} h = 0 < 1 \le \overline{\int_0^1} h.$$

Therefore h is not Riemann integrable. □

7.2-8 Suppose that f is continuous on $[a, b]$, that $f(x) \geq 0$ for all $x \in [a, b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution. Suppose $f(x_0) > 0$ for some $x_0 \in [a, b]$. By the continuity of f, there is a nondegenerate subinterval $[c, d] \subseteq [a, b]$ such that $x_0 \in [c, d]$ and $f(x) > f(x_0)/2$ for all $x \in [c, d]$. Now if P is a partition of $[a, b]$ with $[c, d]$ as a subinterval, we have

$$L(f, P) \geq \frac{f(x_0)}{2}(d - c) =: m > 0.$$

Since $\int_a^b f = 0$, we have

$$0 = \int_a^b f = \underline{\int_a^b} f \geq m > 0,$$

which is a contradiction. Therefore, $f(x) = 0$ for all $x \in [a, b]$. $\square$