MATH 2068 Honours Mathematical Analysis II
2024-25 Term 2
Suggested Solution to Homework 3

6.3-3 Let $f(x) \coloneqq x^2 \sin(1/x)$ for $0 < x \le 1$ and $f(0) \coloneqq 0$, and let $g(x) \coloneqq x^2$ for $x \in [0,1]$. Then both f and g are differentiable on $[0,1]$ and $g(x) > 0$ for $x \neq 0$. Show that $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$ and that $\lim_{x\to 0} f(x)/g(x)$ does not exist.

Solution. We only check that f is differentiable at 0. Indeed, for any $x \in (0,1]$, $\left|\frac{f(x)-f(0)}{x-0}\right| = |x||\sin(1/x)| \le |x|$, so that $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$ by squeeze theorem.

It also follows from squeeze theorem that $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$ because $|f(x)| \le |g(x)| = x^2$ for any $x \in [0,1]$. However, $\lim_{x\to 0} f(x)/g(x) = \lim_{x\to 0} \sin(1/x)$ does not exist, which can be shown by sequential criterion. $\square$

6.4-3 Use Induction to prove Leibniz's rule for the nth derivative of a product:

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x).$$

Solution. The formula is true when $n = 1$ since it is just the product rule. Suppose the formula is true for some $n \in \mathbb{N}$. Differentiating the formula once more, we have

$$\begin{aligned}
&(fg)^{(n+1)}(x)\\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(n-k+1)}(x) g^{(k)}(x) + f^{(n-k)}(x) g^{(k+1)}(x) \right)\\
&= f^{(n+1)}(x)g(x) + \sum_{k=1}^{n} \binom{n}{k} f^{(n-k+1)}(x) g^{(k)}(x) + \sum_{k=0}^{n-1} \binom{n}{k} f^{(n-k)}(x) g^{(k+1)}(x) + f(x)g^{(n+1)}(x)\\
&= f^{(n+1)}(x)g(x) + \sum_{k=1}^{n} \binom{n}{k} f^{(n-k+1)}(x) g^{(k)}(x) + \sum_{k=1}^{n} \binom{n}{k-1} f^{(n-k+1)}(x) g^{(k)}(x) + f(x)g^{(n+1)}(x)\\
&= f^{(n+1)}(x)g(x) + \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) f^{(n-k+1)}(x) g^{(k)}(x) + f(x)g^{(n+1)}(x)\\
&= f^{(n+1)}(x)g(x) + \sum_{k=1}^{n} \binom{n+1}{k} f^{(n+1-k)}(x) g^{(k)}(x) + f(x)g^{(n+1)}(x)\\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1-k)}(x) g^{(k)}(x).
\end{aligned}$$

It follows from Mathematical Induction that the formula holds for all $n \in \mathbb{N}$. $\square$

6.4-4 Show that if $x > 0$, then $1 + \frac{1}{2}x - \frac{1}{8}x^2 \le \sqrt{1+x} \le 1 + \frac{1}{2}x$.

Solution. Let $f(x) = \sqrt{1+x}$. Then, for any $x > -1$,

$$f'(x) = \frac{1}{2\sqrt{1+x}}, \quad f''(x) = -\frac{1}{4(1+x)^{3/2}}, \quad f'''(x) = \frac{3}{8(1+x)^{5/2}}.$$

Fix $x > 0$. By Taylor's Theorem, there exists $c_1 \in (0, x)$ such that

$$\begin{aligned} f(x) &= f(0) + f'(0)(x-0) + \frac{f''(c_1)}{2!}(x-0)^2 \\ &= 1 + \frac{1}{2}x - \frac{1}{8(1+c_1)^{3/2}}x^2. \end{aligned}$$

Since $-\frac{1}{8(1+c_1)^{3/2}}x^2 < 0$, we have $\sqrt{1+x} \leq 1 + \frac{1}{2}x$.

Similarly, there exists $c_2 \in (0, x)$ such that

$$\begin{aligned} f(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(c_2)}{3!}(x-0)^3 \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16(1+c_2)^{5/2}}x^3. \end{aligned}$$

Since $\frac{1}{16(1+c_2)^{5/2}}x^3 > 0$, we have $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x}$. $\square$