MATH 2068 Honours Mathematical Analysis II
2024-25 Term 2
Suggested Solution to Homework 2

6.2-5 Let $a > b > 0$ and let $n \in \mathbb{N}$ satisfy $n \geq 2$. Prove that $a^{1/n} - b^{1/n} < (a-b)^{1/n}$. [*H*int: Show that $f(x) := x^{1/n} - (x-1)^{1/n}$ is decreasing for $x \geq 1$, and evaluate f at 1 and a/b.]

Solution. Let $f(t) = t^{1/n} - (t-1)^{1/n}$ for $t \geq 1$. Then

$$f'(t) = \frac{1}{n}t^{1/n-1} - \frac{1}{n}(t-1)^{1/n-1} < 0 \qquad \text{for } t > 1.$$

For $x > 1$, since f is continuous on $[1, x]$ and differentiable on $(1, x)$, the Mean Value Theorem infers that there is $c_x \in (1, x)$ such that

$$f(x) - f(1) = f'(c_x)(x-1),$$

and so $f(x) < f(1) = 1$. Putting $x = \frac{a}{b} > 1$, we have $f(\frac{a}{b}) < 1$, which yields

$$a^{1/n} - b^{1/n} < (a-b)^{1/n}.$$

$\square$

6.2-7 Use the Mean Value Theorem to prove that $(x-1)/x < \ln x < x - 1$ for $x > 1$. [*H*int: Use that fact that $D \ln x = 1/x$ for $x > 0$.]

Solution. Let $f(t) = \ln t$. Fix $x > 1$. Then f is continuous on $[1, x]$ and differentiable on $(1, x)$ with $f'(t) = 1/t$. By the Mean Value Theorem, there exists $c \in (1, x)$ such that

$$\frac{f(x) - f(1)}{x-1} = f'(c),$$

that is,

$$\ln x = \frac{x-1}{c}.$$

Since $1 < c < x$, it follows that $1/x < 1/c < 1$, and hence

$$\frac{x-1}{x} < \ln x < x - 1.$$

$\square$

6.2-8 Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that if $\lim\limits_{x \to a} f'(x) = A$, then $f'(a)$ exists and equals A. [*Hint*: Use the definition of $f'(a)$ and the Mean Value Theorem.]

Solution. Let $\varepsilon > 0$. Since $\lim\limits_{x \to a} f'(x) = A$, there is $\delta > 0$ such that

$$|f'(x) - A| < \varepsilon \quad \text{ whenever } x \in [a, b] \text{ and } a < x < a + \delta.$$

Suppose $x \in [a, b]$ and $a < x < a + \delta$. By the Mean Value Theorem, there exists $c_x \in (a, x)$ such that

$$f'(c_x) = \frac{f(x) - f(a)}{x - a}.$$

Since $a < c_x < x < a + \delta$, we have

$$\left| \frac{f(x) - f(a)}{x - a} - A \right| = |f'(c_x) - A| < \varepsilon.$$

Therefore, $f'(a)$ exists and equals A. $\square$