MATH 2068 Honours Mathematical Analysis II
2024-25 Term 2
Suggested Solution to Homework 1

6.1-4 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^2$ for x rational, $f(x) := 0$ for x irrational. Show that f is differentiable at $x = 0$, and find $f'(0)$.

Solution. Note that

$$\frac{f(x) - f(0)}{x - 0} = \begin{cases} \frac{x^2 - 0}{x} = x & \text{if } x \in \mathbb{Q}\backslash\{0\}, \\ \frac{0-0}{x} = 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

So,

$$\left| \frac{f(x) - f(0)}{x - 0} \right| \le |x| \quad \text{for any } x \in \mathbb{R}\backslash\{0\}.$$

Since $\lim\limits_{x\to 0} |x| = 0$, it follows from the Squeeze Theorem that $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$. Therefore f is differentiable at $x = 0$ and $f'(0) = 0$. □

6.1-6 Let $n \in \mathbb{N}$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^n$ for $x \ge 0$ and $f(x) := 0$ for $x < 0$. For which values of n is f' continuous at 0? For which values of n is f' differentiable at 0?

Solution. Note that $\lim\limits_{x\to 0^+} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0^+} x^{n-1}$ is 0 when $n \ge 2$; and it does not exist when $n = 1$. Clearly $\lim\limits_{x\to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0^-} 0 = 0$. Hence, $f'(0)$ does not exist when $n = 1$; and when $n \ge 2$

$$f'(x) = \begin{cases} nx^{n-1} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

It is easy to see that f' is continuous for $n \ge 2$.

A similar argument shows that f' is differentiable for $n \ge 3$. □

6.1-13 If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, show that

$$f'(c) = \lim(n\{f(c + 1/n) - f(c)\}).$$

However, show by example that the existence of the limit of this sequence does not imply the existence of $f'(c)$.

Solution. Since f is differentiable at c, we have $\lim\limits_{h\to 0} \dfrac{f(c + h) - f(c)}{h} = f'(c)$. Clearly $\{1/n\}$ is a non-zero sequence with limit 0. By the sequential criterion for limits of functions, we have

$$f'(c) = \lim \frac{f(c + 1/n) - f(c)}{1/n} = \lim(n\{f(c + 1/n) - f(c)\}).$$

For a counterexample, one may consider the Dirichlet function: $f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$

Then $\lim(n\{f(0 + 1/n) - f(0)\}) = \lim(0) = 0$. However, $f'(0)$ does not exist because f is not even continuous at 0.

□