

Topic #15 Tests for Non-absolute Convergence

Keywords:

- Alternating Series Test
- Dirichlet's Test
- Abel's Test

Def. (Alternating Series)

$\sum x_n$ is an alternating series if the terms $(-1)^{n+1}x_n$, $n \in \mathbb{N}$, are all positive (or all negative).

Thm (Alternating Series Test)

An alternating series $\sum (-1)^{n+1}z_n$ with $\begin{cases} \bullet z_n > 0, \forall n \in \mathbb{N} \\ \bullet (z_n) \text{ is decreasing} \end{cases}$ such that $\lim_{n \rightarrow \infty} z_n = 0$ is convergent

Pf: Consider the partial sum S_{2n} :

$$S_{2n} = (z_1 - z_2) + (z_3 - z_4) + \dots + (z_{2n-1} - z_{2n})$$

Note: ① $\because (z_n)$ is decreasing

$$\therefore z_k - z_{k+1} \geq 0, \forall k \in \mathbb{N}$$

then (S_{2n}) is increasing

② Rewrite S_{2n} as

$$S_{2n} = z_1 - (z_2 - z_3) - \dots - (z_{2n-2} - z_{2n-1}) - z_{2n}$$

$$\therefore S_{2n} \leq z_1, \forall n \in \mathbb{N}$$

Monotone Convergence Thm gives:

$$\exists s \in \mathbb{R} \text{ s.t. } s = \lim_{n \rightarrow \infty} S_{2n}$$

To show $\lim_{n \rightarrow \infty} S_n = s$, it suffices to show:

$$\lim_{n \rightarrow \infty} S_{2n+1} = s$$

In fact,

$$\lim_{n \rightarrow \infty} S_{2n+1} = \lim_{n \rightarrow \infty} (S_{2n} + z_{2n+1})$$

$$= \lim_{n \rightarrow \infty} S_{2n} + \lim_{n \rightarrow \infty} z_{2n+1} = s + 0 = s. \quad \#$$

Example:

Recall: p -series $\sum \frac{1}{n^p}$ $\begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$

By Alternating Series Test

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ is convergent if $p > 0$.

Therefore

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ is $\begin{cases} \text{conditionally convergent if } 0 < p \leq 1 \\ \text{absolutely convergent if } p > 1 \\ \text{divergent if } p \leq 0. \end{cases}$

Thm (Dirichlet's Test)

$\sum x_n y_n$ with $\begin{cases} \bullet (x_n) \text{ is decreasing such that } \lim_{n \rightarrow \infty} x_n = 0 \\ \bullet \text{ partial sums } (S_n) \text{ of } \sum y_n \text{ are bounded,} \\ \text{i.e. } \exists B \in \mathbb{R} \text{ s.t. } |S_n| \leq B, \forall n \in \mathbb{N} \end{cases}$
is convergent

pf: claim (Abel's lemma) (or partial summation formula):

$$\forall m > n, \quad \sum_{k=n+1}^m x_k y_k = (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k.$$

$$\begin{aligned} \text{In fact, } \sum_{k=n+1}^m x_k y_k &= \sum_{k=n+1}^m x_k (S_k - S_{k-1}) \\ &= \sum_{k=n+1}^m x_k S_k - \sum_{k=n+1}^m x_k S_{k-1} \\ &= \left(\sum_{k=n+1}^{m-1} x_k S_k + x_m S_m \right) - \left(x_{n+1} S_n + \sum_{k=n+2}^m x_k S_{k-1} \right) \\ &= (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k \\ &\quad \text{(note: } n+2 \leq k \leq m \Rightarrow n+1 \leq k-1 = k' \leq m-1 \text{)} \end{aligned}$$

Therefore,

$$\left| \sum_{k=n+1}^m x_k y_k \right| \leq (x_m + x_{n+1}) B + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) B$$

(note: $x_k - x_{k+1} \geq 0$)

$$= [(x_m + x_{n+1}) + (x_{n+1} - x_m)] B$$

$$= 2x_{n+1} B \rightarrow 0 \text{ as } n \rightarrow \infty$$

Cauchy Convergence Criterion implies:

$\sum x_k y_k$ is convergent. ##

Thm (Abel's Test)

$\sum x_n y_n$ is convergent if

- (x_n) is monotone and convergent
- $\sum y_n$ is convergent.

Pf: Case: (x_n) is decreasing with $x = \lim_{n \rightarrow \infty} x_n$

Define $u_n = x_n - x$

then (u_n) is decreasing with $\lim_{n \rightarrow \infty} u_n = 0$

Write:

$$\sum x_n y_n = \sum (x + u_n) y_n = x \sum y_n + \sum u_n y_n$$

where • $\sum y_n$ is convergent

• $\sum u_n y_n$ is convergent by Dirichlet's Test

(note: $\because \sum y_n$ is convergent

$\therefore$ partial sums of $\sum y_n$ are bounded)

Therefore, $\sum x_n y_n$ is convergent

Case: (x_n) is increasing with $x = \lim_{n \rightarrow \infty} x_n$

Define $u_n = x - x_n$, so $\sum x_n y_n = \sum (x - u_n) y_n$

$$= x \sum y_n - \sum u_n y_n,$$

The rest is similar. ##

Examples (Apply Dirichlet's Test)

Assume: (a_n) is decreasing with $\lim_{n \rightarrow \infty} a_n = 0$

(a) $\sum_{n=1}^{\infty} a_n \cos(nx)$ is convergent if $x \neq 2k\pi$ ($k \in \mathbb{N}$).

Pf: Note:

$$2(\sin \frac{1}{2}x) (\cos x + \dots + \cos nx) = \sin(n+\frac{1}{2})x - \sin \frac{1}{2}x$$

(Exercise)

Then, for $x \neq 2k\pi$ ($k \in \mathbb{N}$),

$$|\cos x + \dots + \cos nx| = \left| \frac{\sin(n+\frac{1}{2})x - \sin \frac{1}{2}x}{2 \sin \frac{1}{2}x} \right|$$

$$\leq \frac{1}{|\sin \frac{1}{2}x|}$$

$\therefore \left(\sum_{k=1}^n \cos kx \right)$ is bounded.

Dirichlet's Test gives:

$\sum_{n=1}^{\infty} a_n \cos nx$ is convergent. #

(b) $\sum_{n=1}^{\infty} a_n \sin nx$ is also convergent if $x \neq 2k\pi$ ($k \in \mathbb{N}$)

(Of course, it is trivial to see:

if $x = 2k\pi$ with $k \in \mathbb{N}$,

then $\sum_{n=1}^{\infty} a_n \sin nx = 0$ is also convergent.)

Pf: Note (again exercise):

$$2(\sin \frac{1}{2}x) (\sin x + \dots + \sin nx) = \cos \frac{1}{2}x - \cos(n+\frac{1}{2})x$$

Then, for $x \neq 2k\pi$ ($k \in \mathbb{N}$),

$$|\sin x + \dots + \sin nx| \leq \frac{1}{|\sin \frac{1}{2}x|}$$

Dirichlet's Test applies. #