# Topic #13 Absolute Convergence

**Def:** Let $(x_n)$ be a sequence in $\mathbb{R}$,

- $\sum x_n$ is **absolutely convergent** if $\sum |x_n|$ is convergent in $\mathbb{R}$;
- $\sum x_n$ is **conditionally convergent** if $\sum x_n$ is convergent but not absolutely converg.

**Thm** $\sum x_n$ is absolutely convergent $\Rightarrow$ $\sum x_n$ is convergent

**Pf:** Let $\epsilon > 0$.

$\because$ $\sum x_n$ is absolutely convergent,

i.e. $\sum |x_n|$ is convergent

$\therefore$ $\exists M(\epsilon) \in \mathbb{N}$ s.t. if $m > n \geq M(\epsilon)$, then

$$\sum_{k=n+1}^{m} |x_k| < \epsilon$$

Letting $S_n = \sum_{k=1}^{n} x_k$, then $\forall m > n \geq M(\epsilon)$,

$$|S_m - S_n| = \left| \sum_{k=n+1}^{m} x_k \right| \leq \sum_{k=n+1}^{m} |x_k| < \epsilon$$

Cauchy's criterion implies that $\sum x_n$ is convergent. #

**Grouping of series:**

$$\sum x_n = x_1 + x_2 + \cdots + x_n + \cdots$$

$$= (x_1 + \cdots + x_{k_1}) \rightsquigarrow \overset{\text{def.}}{=} y_1$$

$$+ (x_{k_1+1} + \cdots + x_{k_2}) \rightsquigarrow \overset{\text{def.}}{=} y_2$$

$$+ \cdots \qquad \vdots$$

( • order of terms fixed

• insert parentheses that group finite terms )

obtain: $(y_k)$.

**Thm:** Let $L = \sum x_n \in \mathbb{R}$ be convergent, then the grouped series $(y_k)$ is convergent and it converges to $L$.

**Pf.** Set $S_n = \sum_{k=1}^{n} x_i$ to be the partial sum of $\sum x_n$

$\because$ $\sum x_n = L$ is convergent

$\therefore$ $\lim_{n\to\infty} S_n = L$

i.e. the sequence $(S_n)$ converges to $L$.

Let $t_k = \sum_{i=1}^{k} y_i$ be the $k^{th}$ partial sum of the grouped series $\sum y_k$.

then $(t_k)$ is a subsequence of $(S_n)$

$\therefore$ $\lim_{k\to\infty} t_k = L$

i.e. $\sum y_k$ converges to $L$. #

**Def.** (Rearrangement of Series)
$\sum y_k$ is a rearrangement of $\sum x_n$
if $\exists$ a bijection $f: \mathbb{N} \to \mathbb{N}$ s.t. $y_k = x_{f(k)}$, $\forall k \in \mathbb{N}$.

**RK.** Rearrangement may affect convergence.
e.g. (Riemann)
Let $\sum x_n$ be conditionally convergent and $c \in \mathbb{R}$,
then $\exists$ a rearrangement of $\sum x_n$ that converges to $c$.

**Thm** Let $\sum x_n$ be absolutely convergent in $\mathbb{R}$,
then any rearrangement $\sum y_k$ of $\sum x_n$ converges to the same value.

**Pf:** Let $L = \sum x_n \in \mathbb{R}$, and let $\epsilon > 0$.

$\because$ $\sum x_n$ is absolutely convergent

$\therefore$ $\exists N \in \mathbb{N}$ s.t. if $n > N$ and $q > N$, then

$$|S_n - L| < \epsilon \quad \text{and} \quad \sum_{k=N+1}^{q} |x_k| < \epsilon.$$

Consider

$$x_1 = y_{f^{-1}(1)}, \quad x_2 = y_{f^{-1}(2)}, \quad \cdots, \quad x_N = y_{f^{-1}(N)}.$$

Define $M = \max\left(f^{-1}(1), \cdots, f^{-1}(N)\right) \in \mathbb{N}$

then all the terms $x_1, \cdots, x_N$ are contained in the partial sum $t_M \stackrel{\text{def.}}{=} y_1 + y_2 + \cdots + y_M$

$\therefore \forall n > N, \forall m \geq M,$

$$\underbrace{|t_m - s_n|}_{\text{contains finite terms excluding } x_1, \cdots, x_N} \leq \sum_{k=N+1}^{q} |x_k| < \epsilon \quad (\text{for some } q > N)$$

$\therefore \forall m \geq M,$

$$|t_m - L| \leq |t_m - s_n| + |s_n - L| \quad (\text{let } n > N)$$
$$< \epsilon + \epsilon = 2\epsilon$$

$\because \epsilon > 0$ is arbitrary

$\therefore \lim_{m \to \infty} t_m = L$ i.e. $\sum y_k$ converges to $L$. #