Topic#3 L'Hospital's Rules

indeterminate forms:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ 0^0,\ 1^{\infty},\ \infty-\infty$$

for

$$\lim \frac{f(x)}{g(x)},\quad \lim f(x)\cdot g(x),\quad \lim f(x)^{g(x)}$$

$$\lim f(x)-g(x)$$

Simple situation

Thm (~~$\frac{0}{0}$ form~~)

Let
- f, g defined on $[a,b]$;
- $f(a)=g(a)=0$;
- $g(x)\neq 0,\ a<x<b$;
- f, g differentiable at a
- $g'(a)\neq 0$

then

$$\lim_{x\to a+}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Pf.

$$\lim_{x\to a+}\frac{f(x)}{g(x)} = \lim_{x\to a+}\frac{f(x)-f(a)}{g(x)-g(a)}$$

$$= \lim_{x\to a+}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \frac{\lim_{x\to a+}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a+}\frac{g(x)-g(a)}{x-a}}$$

$$= \frac{f'(a)}{g'(a)}.$$

Thm (Cauchy Mean Value Theorem)

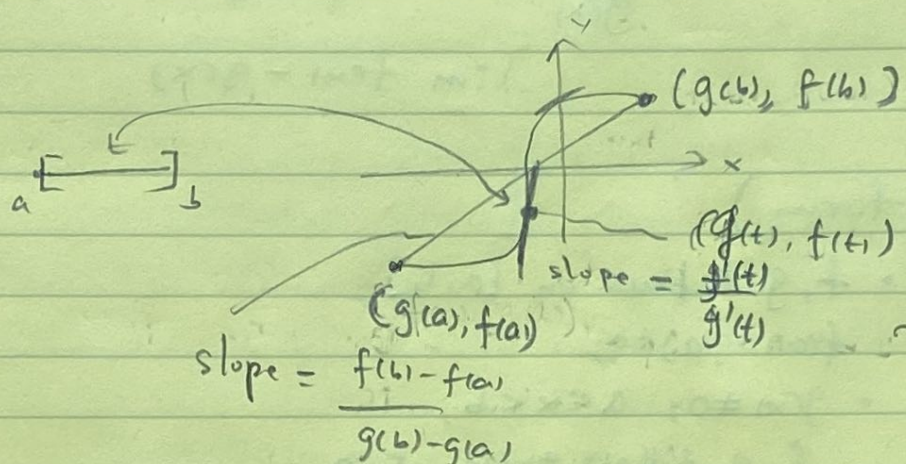
Let • f, g continuous on $[a, b]$

• f, g differentiable on (a, b)

• $g'(x) \neq 0$, $\forall x \in (a, b)$.

then $\exists c \in (a, b)$ s.t.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$



Pf: note by Rolle's Thm that $g(a) \neq g(b)$,

(otherwise $g(a) = g(b)$, then $\exists x_0 \in (a, b)$ s.t. $g'(x_0) = 0$)

Define

$$h(x) = \frac{f(b) - f(a)}{g(b) - g(a)} (g(x) - g(a)) - (f(x) - f(a))$$

then - h continuous on $[a, b]$

- h differentiable on (a, b)

- $h(a) = 0 = h(b)$

so, $\exists c \in (a, b)$ s.t.

$$0 = h'(c) = \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) - f'(c), \quad \#$$

RK: take $g(x) = x$, gives Mean Value Theorem. #

Move it to Topic #2.

Thm (L'Hospital's Rule I; $\frac{0}{0}$ form)

Let • $-\infty \le a < b \le \infty$

• f, g differentiable on (a,b) with $g'(x)$

• $g'(x) \neq 0, \ \forall x \in (a,b)$

• $\lim_{x\to a+} f(x) = 0 = \lim_{x\to a+} g(x)$

then if $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{-\infty, \infty\}$ (exist, either finite or infinite)

then $\lim_{x\to a+} \frac{f(x)}{g(x)} = \lim_{x\to a+} \frac{f'(x)}{g'(x)} = L$.

Pf: Claim (preparation):

Let $a < \alpha < \beta < b$, then By Rolle's Thm, $g(\beta) \neq g(\alpha)$.

By Cauchy Mean Value Thm, $\exists\, u = u_{\alpha,\beta} \in (\alpha, \beta)$ s.t.

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(u)}{g'(u)}.$$

Case (a): $L \in \mathbb{R}$ finite

Let $\epsilon > 0$, Then $\exists\, c \in (a,b)$ s.t.

$$L - \epsilon < \frac{f'(u)}{g'(u)} < L + \epsilon, \quad \forall u \in (a, c)$$

Claim gives

$$L - \epsilon < \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} < L + \epsilon, \quad a < \alpha < \beta \le c$$

Take $\alpha \to a+$, then

$$L - \epsilon < \frac{f(\beta)}{g(\beta)} < L + \epsilon, \quad a < \beta \le c$$

therefore, $\lim_{x\to a+} \frac{f(x)}{g(x)} = L$. #

Case (b): $L = \infty$ infinite (similarly for $L = -\infty$)

Let $M > 0$. Then $\exists\, c \in (a,b)$ s.t.

$$\frac{f'(u)}{g'(u)} > M, \quad \forall u \in (a, c)$$

The claim gives that

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} > M, \quad a<\alpha<\beta \le C$$

Take $\alpha \to a+$,

$$\frac{f(\beta)}{g(\beta)} > M, \quad a<\beta<C$$

$$\therefore \lim_{x\to a+} \frac{f(x)}{g(x)} = \infty, \quad \#$$

examples.

① $$\lim_{x\to 0+} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0+} \frac{\cos x}{\frac{1}{2}x^{-\frac{1}{2}}}$$

$$= \lim_{x\to 0+} 2x^{\frac{1}{2}} \cos x$$

$$= 0$$

② $$\lim_{x\to 0+} \frac{1-\cos x}{x^2} = \lim_{x\to 0+} \frac{\sin x}{2x}$$ twice L'Hospital

$$= \lim_{x\to 0+} \frac{\cos x}{2}$$

$$= \frac{1}{2}$$

③ $$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1$$

④ $$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = 1$$

not really necessary

Thm (L'Hospital's Rule II: $\frac{\infty}{\infty}$-form)

Let
- $-\infty \le a < b \le \infty$.
- f, g differentiable on (a,b)
- $g'(x) \neq 0, \ \forall x \in (a,b)$.
- $\lim_{x \to a+} g(x) = \pm\infty$

Then if $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{\pm\infty\}$. (exists, finite or infinite)

then $$\lim_{x\to a+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{f'(x)}{g'(x)} \quad (=L).$$

Pf: Claim: Let $a < \alpha < \beta < b$, then $g(\beta) \neq g(\alpha)$ and $\exists u \in (\alpha, \beta)$ s.t.

$$\frac{f(\beta) - f(\alpha)}{g'(\beta) - g(\alpha)} = \frac{f'(u)}{g'(u)}.$$

Case (a): $\lim_{x\to a+} g(x) = +\infty$, ($L > 0$ is finite.

(similarly consider $L=0$ or $L<0$)

Let $\epsilon > 0$ be given.

$\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L$ tells: $\left\{ \begin{array}{l} \exists c \in (a,b) \text{ s.t.} \\ L - \epsilon < \frac{f'(u)}{g'(u)} < L + \epsilon, \ \forall u \in (a, c] \end{array} \right.$

Let $\beta = c$, then

$$(0<) L - \epsilon < \frac{f(c) - f(\alpha)}{g(c) - g(\alpha)} < L + \epsilon, \quad \forall \alpha \in (a,c).$$

Multiply $\frac{g(\alpha) - g(c)}{g(\alpha)} = 1 - \frac{g(c)}{g(\alpha)}$ (>0, WLG),

$$(L - \epsilon)\left(1 - \frac{g(c)}{g(\alpha)}\right) < \frac{f(\alpha)}{g(\alpha)} - \frac{f(c)}{g(\alpha)} < (L+\epsilon)\left(1 - \frac{g(c)}{g(\alpha)}\right)$$

Note: $\lim_{\alpha\to a+} \frac{g(c)}{g(\alpha)} = 0$, $\lim_{\alpha\to a+} \frac{f(c)}{g(\alpha)} = 0$,

then $\forall \delta \in (0,1)$, $\exists d \in (a,c)$ s.t.

$$0 < \frac{g(c)}{g(\alpha)} < \delta, \quad \frac{|f(c)|}{g(\alpha)} < \delta, \quad \forall \alpha \in (a,d).$$

$$L - L\delta - \epsilon + \epsilon\delta - \delta$$
$$> L - (L+1)\delta - \epsilon$$

thus for $a < \alpha < d$,

$$(L-\epsilon)(1-\delta) - \delta < \frac{f(\alpha)}{g(\alpha)} < (L+\epsilon) + \delta$$

take $\delta = \min\{1, \epsilon, \frac{\epsilon}{L+1}\}$, then

$$L - 2\epsilon < \frac{f(\alpha)}{g(\alpha)} < L + 2\epsilon, \ \forall \alpha \in (a, d)$$

therefore,

$$\lim_{x \to \alpha+} \frac{f(x)}{g(x)} = L. \ \#$$

Case (b): $\lim_{x \to a+} g(x) = +\infty$, $L = +\infty$.

Let $M > 1$ be given.

similarly, $\exists\, c \in (a, b)$

- $\dfrac{f(c) - f(\alpha)}{g(c) - g(\alpha)} > M$, $a < \alpha < c$
- $g(\alpha) > 0$, $0 < \dfrac{g(c)}{g(\alpha)} < \dfrac{1}{2}$, $a < \alpha < c$.
- $\dfrac{|f(c)|}{g(\alpha)} < \dfrac{1}{2}$, $a < \alpha < c$

then, multiply $\dfrac{g(\alpha) - g(c)}{g(\alpha)} = 1 - \dfrac{g(c)}{g(\alpha)} \in (\frac{1}{2}, 1)$

$$\frac{f(\alpha)}{g(\alpha)} - \frac{f(c)}{g(\alpha)} > M\left(1 - \frac{g(c)}{g(\alpha)}\right) > \frac{1}{2}M$$

so,

$$\frac{f(\alpha)}{g(\alpha)} > \frac{1}{2}M + \frac{f(c)}{g(\alpha)} > \frac{1}{2}(M-1), \ a < \alpha < c.$$

Therefore,

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \infty. \quad \#$$

Other cases can be treated similarly. ##

examples :

① $\lim_{x\to\infty} \frac{\ln x}{x} \overset{(*)}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

$\left[\begin{array}{l} \bullet \lim_{x\to\infty} g(x) = \infty \\ \bullet\ Df = \frac{1}{x},\ \forall x \neq 0 \\ \quad Dg = 1 \neq 0 \\ \bullet \lim_{x\to\infty} \frac{1}{x} = 0 \text{ exists} \end{array}\right.$

② $\lim_{x\to\infty} \frac{x^2}{e^x} \overset{(*)}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{(**)}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$

$(*) \left[\begin{array}{l} \bullet \lim_{x\to\infty} g(x) = \infty \\ \bullet\ Df = 2x,\ Dg = e^x \neq 0,\ \forall x \\ \bullet \lim_{x\to\infty} \frac{Df(x)}{Dg(x)} \text{ exists} \end{array}\right.$

$(**) \left[\begin{array}{l} \bullet \lim_{x\to\infty} g(x) = \infty \\ \bullet\ Df = 2 \\ \quad Dg = e^x \\ \lim_{x\to\infty} \frac{Df(x)}{Dg(x)} = 0 \text{ exists} \end{array}\right.$

③ $\lim_{x\to 0^+} \frac{\ln \sin x}{\ln x} = \lim_{x\to 0^+} \frac{\frac{1}{\sin x}\cos x}{\frac{1}{x}}$

$= \lim_{x\to 0^+} \frac{x}{\sin x} \cdot \cos x$

$= \lim_{x\to 0^+} \frac{x}{\sin x} \cdot \lim_{x\to 0^+} \cos x$

$= 1 \cdot 1 = 1$

④ $\lim_{x\to\infty} \frac{x - \sin x}{x + \sin x} = \lim_{x\to\infty} \frac{1 - \frac{\sin x}{x}}{1 + \frac{\sin x}{x}}$

$= \frac{1-0}{1+0} = 1$

$\cancel{*}\ \lim_{x\to\infty} \frac{1-\cos x}{1+\cos x}$

(Warning)

$\lim_{x\to\infty} g(x)$ does NOT exist !

Other Indetermined forms

$0 \cdot \infty = \frac{\infty}{\frac{1}{0}}$, $0^0 = e^{0 \ln 0} = e^{0 \cdot \infty} = e^{\frac{\infty}{\frac{1}{0}}}$

$1^\infty = e^{0 \cdot \infty}$, $\infty^0 = e^{0 \cdot \infty}$

$\infty - \infty$

examples

① $\lim_{x\to 0+} x\ln x \overset{0\cdot\infty}{=} \lim_{x\to 0+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$

$= \lim_{x\to 0+} (-x) = 0$

② $\lim_{x\to 0+} x^x \overset{0^0}{=} \lim_{x\to 0+} e^{x\ln x} = e^0 = 1$

③ $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \overset{1^\infty}{=\!=} \lim_{x\to\infty} e^{x\ln(1+\frac{1}{x})}$

$= e^{\lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}}$ $\rightarrow \ln\frac{1+x}{x} = \ln(1+x) - \ln x$

$= e^{\lim_{x\to\infty} \frac{\frac{1}{1+x} - \frac{1}{x}}{-\frac{1}{x^2}}}$

$= e^{\lim_{x\to\infty} \frac{(-1)}{(1+x)(x)}(-x^2)}$

$= e^{\lim_{x\to\infty} \frac{1}{1+\frac{1}{x}}}$

$= e$

④ $\lim_{x\to 0+} \left(1+\frac{1}{x}\right)^x \overset{\infty^0}{=} \lim_{x\to 0+} e^{x\ln(1+\frac{1}{x})}$

$= \exp\left(\lim_{x\to 0+} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}\right)$ $\rightarrow \ln(1+x) - \ln x$

$= \exp\left(\lim_{x\to 0+} \frac{\frac{1}{1+x} - \frac{1}{x}}{-\frac{1}{x}}\right)$

$= \exp\left(\lim_{x\to 0+} \frac{1}{1+\frac{1}{x}}\right)$

$= \exp(0) = 1$

⑤ $\lim_{x\to 0+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) \overset{\infty-\infty}{=} \lim_{x\to 0+} \frac{\sin x - x}{x\sin x}$

$\overset{\frac{0}{0}}{=} \lim_{x\to 0+} \frac{\cos x - 1}{\sin x + x\cos x} \overset{\frac{0}{0}}{=} \lim_{x\to 0+} \frac{-\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$ #

examples

① $\lim_{x\to 0+} x\ln x \overset{0\cdot\infty}{=} \lim_{x\to 0+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$

$= \lim_{x\to 0+} (-x) = 0$

② $\lim_{x\to 0+} x^x \overset{0^0}{=} \lim_{x\to 0+} e^{x\ln x} = e^0 = 1$

③ $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \overset{1^\infty}{=} \lim_{x\to\infty} e^{x\ln(1+\frac{1}{x})}$

$\ln\frac{1+x}{x} = \ln(1+x) - \ln x$

$= e^{\lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}}$

$= e^{\lim_{x\to\infty} \frac{\frac{1}{1+x} - \frac{1}{x}}{-\frac{1}{x^2}}}$

$= e^{\lim_{x\to\infty} \frac{(-1)}{(1+x)(x)}(-x^2)}$

$= e^{\lim_{x\to\infty} \frac{1}{1+\frac{1}{x}}}$

$= e$

④ $\lim_{x\to 0+} \left(1+\frac{1}{x}\right)^x \overset{\infty^0}{=} \lim_{x\to 0+} e^{x\ln(1+\frac{1}{x})}$

$= \exp\left(\lim_{x\to 0+} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}\right)$ $\ln(1+x) - \ln x$

$= \exp\left(\lim_{x\to 0+} \frac{\frac{1}{1+x} - \frac{1}{x}}{-\frac{1}{x}}\right)$

$= \exp\left(\lim_{x\to 0+} \frac{1}{1+\frac{1}{x}}\right)$

$= \exp(0) = 1$

⑤ $\lim_{x\to 0+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) \overset{\infty-\infty}{=} \lim_{x\to 0+} \frac{\sin x - x}{x\sin x}$

$\overset{\frac{0}{0}}{=} \lim_{x\to 0+} \frac{\cos x - 1}{\sin x + x\cos x} \overset{\frac{0}{0}}{=} \lim_{x\to 0+} \frac{-\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$ #

L'Hospital's Rule:

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} \;\underset{\text{Case (II): } \lim_{x\to a^+} g(x)=\infty}{\overset{\text{Case (I): } \lim_{x\to a^+} f(x)=0,\ \lim_{x\to a^+} g(x)=0}{=\!=\!=}}\; \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

Provided that the RHS limit <u>exists</u>
[finite or infinity ∞ or $-\infty$]

and $\exists$ a deleted neighb $\mathring{U}_a = (a, a+\delta)$

s.t. ① f, g differentiable in $\mathring{U}_a$

② $g'(x) \neq 0$, $\forall x \in \mathring{U}_a$.