Topic#1 Derivative

Def: $f: \underset{\substack{\nearrow \subseteq \mathbb{R} \\ \text{interval}}}{\overset{c\in}{I}} \longrightarrow \mathbb{R}$

(1) f is differentiable at c if the limit

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

exists, namely, $\exists\, L \in \mathbb{R}$ s.t. $L = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$

i.e. $\forall\, \epsilon > 0 \quad \exists\, \delta(\epsilon) > 0$ s.t.

if $x \in I$ with $0 < |x-c| < \delta(\epsilon)$

then $\left| \frac{f(x)-f(c)}{x-c} - L \right| < \epsilon$.

(2) The limit value L is said to be the derivative of f at c, usually denoted by

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}. \quad \#$$

Rk: We unstand f' as a function:

$$f' : D \overset{\text{def.}}{=} \{x \in I : f \text{ is differentiable at } x\} \longrightarrow \mathbb{R}$$

Thm: If $f: I \to \mathbb{R}$ is differentiable at $c \in I$ then f is continuous at c

Pf:

$$\lim_{x\to c}\left[f(x)-f(c)\right] = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot (x-c)$$

$$\overset{\left(\substack{f \text{ is differentiable} \\ \text{at } c\in I}\right)}{=} \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot \lim_{x\to c} (x-c)$$

$$= f'(c) \cdot 0 = 0. \quad \#$$

RK: $\exists$ a function which is continuous everywhere but differentiable nowhere. (Karl Weierstrass 1872)

Thm: Let $f, g : I \to \mathbb{R}$ be differentiable at $c \in I$, then

(a) $(\alpha f)'(c) = \alpha f'(c)$, $\alpha \in \mathbb{R}$

(b) $(f+g)'(c) = f'(c) + g'(c)$

(c) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

(d) $$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2} \qquad (g(c) \neq 0)$$

RK: Thm tells

αf, $f+g$, fg, $\frac{f}{g}$ $(g(c) \neq 0)$

are differentiable at $c \in I$.

Pf: prove (c) only:

Let $x \in I$ with $x \neq c$,

$$\frac{(fg)(x) - (fg)(c)}{x-c} = \frac{f(x)g(x) - f(c)g(c)}{x-c}$$

$$= \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x-c}$$

$$= \frac{f(x)-f(c)}{x-c} \cdot g(x) + f(c) \cdot \frac{g(x)-g(c)}{x-c}$$

Hence

$$\lim_{x\to c} (\ldots) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot \lim_{x\to c} g(x) + f(c) \cdot \lim_{x\to c} \frac{g(x)-g(c)}{x-c}$$

$$= f'(c)g(c) + f(c)g'(c). \quad \#$$

Hint to prove (d): Let $x \in I$ with $x \neq c$:

$$\frac{\left(\frac{1}{g}\right)(x) - \left(\frac{1}{g}\right)(c)}{x-c} = \frac{\frac{1}{g(x)} - \frac{1}{g(c)}}{x-c}$$

$$= \frac{g(c) - g(x)}{g(x)\,g(c)\,(x-c)}$$

$$= -\frac{1}{g(x)g(c)} \; \frac{g(x)-g(c)}{x-c}$$

$$\lim_{x\to c} (\dots) = -\left[\lim_{x\to c} \frac{1}{g(x)}\right] \cdot \frac{1}{g(c)} \cdot \lim_{x\to c} \frac{g(x)-g(c)}{x-c}$$

$$= -\frac{1}{g(c)} \cdot \frac{1}{g(c)} \cdot g'(c)$$

$$= -\frac{g'(c)}{(g(c))^2}$$

So, $\frac{1}{g}$ is differentiable at $c \in \mathbb{R}$ with

$$\left(\frac{1}{g}\right)'(c) = -\frac{g'(c)}{(g(c))^2}$$

Then, apply (c) to $\frac{f}{g} = f \cdot \frac{1}{g}$. #

<u>Coro</u>: Let $f_1, f_2, \dots, f_n : I \to \mathbb{R}$ be differentiable at $c \in I \subseteq \mathbb{R}$, then

(a) $(f_1 + f_2 + \dots + f_n)'(c) = f_1'(c) + f_2'(c) + \dots + f_n'(c)$

(b) $$(f_1 f_2 \cdots f_n)'(c) = f_1'(c) f_2(c) \cdots f_n(c) + f_1(c) f_2'(c) \cdots f_n(c) + \dots + f_1(c) f_2(c) \cdots f_n'(c).$$

In particular, if $f_1 = \dots = f_n = f$, then

$$(f^n)'(c) = n\,(f(c))^{n-1} f'(c).$$ #

(derivative as a function)

Note: Sometimes, write f' as Df, so

$$D(f+g) = Df + Dg$$
$$D(fg) = (Df)\cdot g + f\cdot(Dg)$$

Chain Rule:
derivative of a composition function
$$g\circ f(x) = g(f(x))\ ?$$

Lemma (Carathéodory's Thm)
$f: I\to\mathbb{R}$ is differentiable at $c\in I$
iff $\exists\ \varphi: I\to\mathbb{R}$ which is continuous at c such that
$$f(x) - f(c) = \varphi(x)(x-c),\ \forall x\in I.$$

In this case, $\varphi(c) = f'(c)$.

Pf. ($\Rightarrow$) Let $f: I\to\mathbb{R}$ be differentiable at $c\in I$.
Define
$$\varphi(x) = \begin{cases} \dfrac{f(x)-f(c)}{x-c} & \text{for } x\in I \text{ with } x\neq c \\ f'(c) & \text{for } x = c \end{cases}$$

(a) $\varphi: I\to\mathbb{R}$ is well-defined and continuous at c.
Indeed, $\lim\limits_{x\to c}\varphi(x) = \lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c} = f'(c) = \varphi(c)$.

(b) Obvious to see: $f(x) - f(c) = \varphi(x)(x-c),\ \forall x\in I$.

($\Leftarrow$) $\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c} = \lim\limits_{x\to c}\dfrac{\varphi(x)(x-c)}{x-c}$

$= \lim\limits_{x\to c}\varphi(x)$

$= \varphi(c)$ ($\because \varphi$ continuous at c).

#

Thm (Chain Rule)

Let $f: I \to \mathbb{R}$ and $g: J \to \mathbb{R}$ be functions such that $f(I) \subseteq J$. If f is differentiable at $c \in I$ and g is differentiable at $f(c) \in J$ then $g \circ f$ is differentiable at c with

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

Pf: $\because$ f differentiable at $c \in I$

$\therefore \exists \varphi: I \to \mathbb{R}$ continuous at c, with $\varphi(c) = f'(c)$ such that

$$f(x) - f(c) = \varphi(x)(x-c), \quad \forall x \in I$$

similarly,

$\exists \psi: J \to \mathbb{R}$ continuous at $f(c)$ with $\psi(f(c)) = g'(f(c))$

such that $g(y) - g(f(c)) = \psi(y)(y - f(c)), \forall y \in J$.

Therefore, let $x \in I$, then

$$\begin{aligned} & g(f(x)) - g(f(c)) \\ &= \psi(f(x))(f(x) - f(c)) \\ &= \psi(f(x))\varphi(x)(x-c), \\ &= [\psi \circ f(x)\, \varphi(x)](x-c), \end{aligned}$$

where $(\psi \circ f) \cdot \varphi : I \to \mathbb{R}$ is continuous at c,

Thus, by Carathéodory Thm, $g \circ f$ is differentiable at $c \in I$ with

$$\begin{aligned} (g \circ f)'(c) &= \psi \circ f(c)\, \varphi(c) \\ &= g'(f(c)) \cdot f'(c). \end{aligned}$$

#

Differentiability of Inverse Functions

Let $f: I \to J \overset{\text{def.}}{=} f(I) \subset \mathbb{R}$ then $g(f(x)) = x$

$g \overset{\text{def.}}{=} f^{-1}: J \to I$

$$g'(f(c))f'(c) = 1$$

$$g'(f(c)) = \frac{1}{f'(c)}$$

if f diff at c, g diff at $f(c)$, $f'(c) \neq 0$. ✱

Thm Let both

$f: I \to \mathbb{R}$ and its invers $g \overset{\text{def}}{=} f^{-1}: J \overset{\text{def}}{=} f(I) \to \mathbb{R}$ be strictly monotone and continuous. If f is differentiable at $c \in I$ with $f'(c) \neq 0$, then g is differentiable at $d \overset{\text{def}}{=} f(c)$ with

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}$$

Pf. $\because$ f differentiable at c

$\therefore$ $\exists \varphi: I \to \mathbb{R}$, continuous at c, such that and $\varphi(c) = f'(c)$

$$f(x) - f(c) = \varphi(x)\,(x-c), \ \forall x \in I$$

$\because$ $\varphi(c) = f'(c) \neq 0$, φ continuous at c

$\therefore$ $\exists V \overset{\text{def}}{=} (c-\delta, c+\delta)$ s.t. $\varphi(x) \neq 0$, $\forall x \in V \cap I$

Define $U = f(V \cap I)$, then $\forall y \in U$,

$$f(g(y)) = y, \quad \forall y \in U$$

so

$$\begin{aligned} y - d &= f(g(y)) - f(c) \\ &= \varphi(g(y)) \cdot (g(y) - c) \\ &= \varphi(g(y)) \cdot (g(y) - g(d)) \end{aligned}$$

Since $\varphi(g(y)) \neq 0$ for $y \in U$, it holds:

$$g(y) - g(d) = \frac{1}{\varphi(g(y))}\,(y-d), \ \forall y \in U$$

where $\frac{1}{\varphi \circ g}$ is continuous at $d \in U$.

by Carathéodory Thm,

Therefore, g is differentiable at d with

$$g'(d) = \frac{1}{\varphi(g(d))} = \frac{1}{\varphi(c)} = \frac{1}{f'(c)}. \quad \#$$

RK: Assumption $f'(c) \neq 0$ is essential. Example: $f(x) = x^3$ with $c = 0$

$g(y) = y^{\frac{1}{3}}$, $d = f(0) = 0$, g is NOT differentiable at $y = 0$.

Corollary: Let $f: I \to \mathbb{R}$ be strictly monotone and $g = f^{-1}: J \overset{\text{def.}}{=} f(I) \to \mathbb{R}$ be the inverse to f. If f is differentiable on I with $f'(x) \neq 0$ for any $x \in I$, then g is differentiable on J with

$$g' = \frac{1}{f' \circ g} \quad \text{on } J,$$

$$\text{i.e.} \quad g'(y) = \frac{1}{f'(g(y))}, \quad \forall y \in J.$$

Pf. Let f be differentiable on I, then f is continuous on I, and by the Continuous Inverse Thm, the inverse $g = f^{-1}$ is also strictly monotone and continuous on $J = f(I)$.

#

e.g. 1 $f(x) = x^n$ $(n \in \mathbb{N})$, $x \in [0, \infty) \overset{\text{def.}}{=} I$

Let $y = f(x) = x^n$, then $x = y^{\frac{1}{n}}$, so

$g(y) = f^{-1}(y) = y^{\frac{1}{n}}$, $y \in [0, \infty) = I$

$f'(x) = nx^{n-1}$, $x \in [0, \infty)$; $\quad f'(x) > 0$, $x \in (0, \infty)$

For $y > 0$, g is differentiable at y with

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{n\left(y^{\frac{1}{n}}\right)^{n-1}} = \frac{1}{n y^{\frac{n-1}{n}}}$$

$$= \frac{1}{n} y^{\frac{1}{n} - 1}.$$

#

e.g. 2 $f(x) \overset{\text{def.}}{=} \sin x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \overset{\text{def.}}{=} I$:

strictly monotone, differentiable on I

$f'(x) = \cos x$; $\quad f'(x) > 0$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

the inverse

$$g(y) = f^{-1}(y) = \operatorname{Arcsin} y$$

is differentiable on $(-1, 1)$ with

$x = g(y)$

$y = \sin x$

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - \sin^2 x}} = \frac{1}{\sqrt{1 - y^2}}$$

#