Counterexamples for Dini's Theorem@2025March14

Let's recall:

Theorem 0.1 (Dini's Theorem)**.** *Let $(f_n(x))_{n\geq 1}$ be a monotone sequence of continuous functions on the compact (equivalently, bounded and closed) interval $I := [a, b]$ that converges pointwise to a continuous function f, then the convergence is uniform.*

We check what happens if some of the hypothesis was dropped:

Case I is not compact:

Let $I = (0, 1)$, then I is bounded but not closed, hence not compact. Consider

$$f_n(x) = x^n, \quad 0 < x < 1. \tag{0.1}$$

Then, (f_n) is a decreasing sequence of continuous functions on $(0, 1)$ that converges pointwise to the zero function. The limit function is also continuous on $(0, 1)$. But, the convergence is not uniform, because for any $n \geq 1$,

$$\sup_{0<x<1} x^n = 1. \tag{0.2}$$

Let $I = \mathbb{R}$, then I is closed but not bounded, hence not compact. Consider

$$f_n(x) = \begin{cases} 0 & \text{for } x < n, \\ \dfrac{x-n}{n} & \text{for } n \leq x < 2n, \\ 1 & \text{for } x \geq 2n. \end{cases} \tag{0.3}$$

Then, (f_n) is a decreasing sequence of continuous functions on $\mathbb{R}$ and converges pointwise to 0. But, the convergence is not uniform, because

$$\sup_{x\in\mathbb{R}} f_n(x) = 1. \tag{0.4}$$

Case f is not continuous:

Let $I = [0, 1]$, then I is bounded and closed, hence compact. Consider

$$f_n(x) = \begin{cases} 1 - 2nx & \text{for } 0 \leq x < \dfrac{1}{2n}, \\ 0 & \text{for } \dfrac{1}{2n} \leq x \leq 1. \end{cases} \tag{0.5}$$

Then, (f_n) is a decreasing sequence of continuous functions on $[0, 1]$ and converges pointwise to the discontinuous function f equal to 1 at 0 and to 0 elsewhere. But, the convergence is not uniform, because

$$\sup_{0\leq x\leq 1} |f_n(x) - f(x)| = 1. \tag{0.6}$$

Case (f_n) is not monotone:

Let $I = [0,1]$. Consider

$$f_n(x) = \begin{cases} 0 & \text{for } 0 \le x < \dfrac{1}{2n}, \\ 4nx - 2 & \text{for } \dfrac{1}{2n} \le x < \dfrac{3}{4n}, \\ 4 - 4nx & \text{for } \dfrac{3}{4n} \le x < \dfrac{1}{n}, \\ 0 & \text{for } \dfrac{1}{n} \le x \le 1. \end{cases} \tag{0.7}$$

Then, (f_n) is a sequence of continuous functions on $[0,1]$ and converges pointwise to the zero function, but (f_n) is not monotone. Again, the convergence is not uniform, because

$$\sup_{0 \le x \le 1} f_n(x) = 1. \tag{0.8}$$

Case f_n is not continuous: Let $I = [0,1]$. Consider

$$f_n(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 - \dfrac{1}{n}, \\ 1 & \text{for } 1 - \dfrac{1}{n} \le x < 1, \\ 0 & \text{for } x = 1. \end{cases} \tag{0.9}$$

Then, (f_n) is a decreasing sequence of discontinuous functions on $[0,1]$ and converges pointwise to the zero function. Again, the convergence is not uniform, because

$$\sup_{0 \le x \le 1} f_n(x) = 1. \tag{0.10}$$