Pf of Thm 9.4.10: Since $[a,b] \subset (-R,R)$, $\exists\ 0<c<1$ such that

$-cR < a$ and $b < cR$. (Note: c depends only on a, b)

Therefore $\forall\ x \in [a,b]$, $|x| < cR$.

By argument in the proof of Cauchy-Hadamard Thm, we have

$\exists\ K \in \mathbb{N}$ s.t. $|a_n x^n| \le c^n$, $\forall\ n \ge K$

Since $\sum c^n$ is convergent, Weierstrass M-Test (Thm 9.4.6)

$\Rightarrow$ ($\sum_{n=K}^{\infty} a_n x^n$ and hence) $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[a,b]$. #

Thm 9.4.11

- The limit of power series is continuous on the interval of convergence.
- A power series can be integrated term-by-term over any closed and bounded interval contained in the interval of convergence.

Pf: • $\forall\ x \in (-R,R)$, choose a closed & bounded interval $[a,b]$

s.t. $x \in [a,b] \subset (-R,R)$. Then on $[a,b]$,

$\sum a_n x^n$ converges uniformly. (Thm 9.4.10)

Thm 9.4.2 $\Rightarrow$ $\sum_{n=1}^{\infty} a_n x^n$ is continuous on $[a,b]$ and hence at x

Since $x \in (-R,R)$ is arbitrary, $\sum_{n=0}^{\infty} a_n x^n$ is continuous on $(-R,R)$.

- For any closed and bounded interval $[a,b] \subset (-R,R)$,

  $\sum a_n x^n$ converges uniformly on $[a,b]$

  and hence Thm 9.4.3 $\Rightarrow$ integrability and

$$\int_a^b \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \int_a^b a_n x^n . \quad \divideontimes$$

<u>Thm 9.4.12</u> (<u>Differentiation Thm</u>)

A power series can be <u>differentiated term-by-term</u> within the <u>interval of convergence</u>. In fact, if R = radius of convergence of $\sum a_n x^n$

and $\quad f(x) = \sum_{n=0}^{\infty} a_n x^n$, for $|x| < R$,

then <u>the radius of convergence of $\sum_{n=0}^{\infty} n a_n x^{n-1} = R$</u>,

and $\quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, for $|x| < R$

<u>Pf</u>: All conditions of Thm 8.2.3 on interchanging of limit and derivative are clearly satisfied when restricted to a closed and bounded interval $[a,b] \subset (-R,R)$ (using Thm 9.4.10)

except the uniform convergence of the $\sum (a_n x^n)' = \sum n a_n x^{n-1}$ on $[a,b]$ needs a proof.

By Thm 9.4.10, we only need to prove the following

Radius of convergence of $\sum n a_n x^{n-1} = R$

$= $ Radius of convergence of $\sum a_n x^n$

Pf

Since $n^{\frac{1}{n}} \to 1$, the seq. $\left(|(n+1)a_{n+1}|^{\frac{1}{n}}\right)$ is bounded

$\Leftrightarrow$ the seq $\left(|a_n|^{\frac{1}{n}}\right)$ is bounded

unbounded case:

$R=0 \Leftrightarrow$ Radius of convergence of $\sum n a_n x^{n-1} = 0$

bounded case:

$\left(\text{Radius of convergence of } \sum n a_n x^{n-1}\right)^{-1} = \limsup |(n+1) a_{n+1}|^{\frac{1}{n+1}}$ (check!)

$= \limsup |n a_n|^{\frac{1}{n}} = \limsup \left(n^{\frac{1}{n}} |a_n|^{\frac{1}{n}}\right)$

$= \limsup |a_n|^{\frac{1}{n}}$ (since $n^{\frac{1}{n}} \to 1$)

$= R^{-1}$.

$\therefore$ The claim and hence the Thm is proved since $[a,b] \subset (-R,R)$ is arbitrary.

#

Remarks: (i) Differentiation Thm 9.4.12 makes no conclusion for $|x|=R$:

e.g. $\sum \frac{1}{n^2}x^n$ converges for $|x|=1$ $(=R)$

but $\left(\sum \frac{1}{n^2}x^n\right)' = \sum \frac{1}{n}x^{n-1}$, $\begin{cases} \text{converges at } x=-1 \\ \text{diverges at } x=1. \end{cases}$

(ii) Repeated application of Thm 9.4.12 $\Rightarrow$

$$\forall k \in \mathbb{N}, \quad \boxed{\left(\sum_{n=0}^{\infty} a_n x^n\right)^{(k)} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}} \quad (|x|<R)$$