**Thm 9.2.2** (Root Test) (Cauchy)

(a) If $\exists\, r<1$ and $K \in \mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} \le r, \quad \forall\, n \ge K,$$

then $\sum x_n$ is absolutely convergent.

(b) If $\exists\, K \in \mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} \ge 1, \quad \forall\, n \ge K,$$

then $\sum x_n$ is divergent.

Pf: (a) If $|x_n|^{\frac{1}{n}} \le r$, $\forall\, n \ge K$

then $|x_n| \le r^n$, $\forall\, n \ge K$

Since $\sum r^n$ is convergent for $0 \le r < 1$,

Comparison Test 3.7.7 $\Rightarrow$ $\sum |x_n|$ is convergent.

(b) If $|x_n|^{\frac{1}{n}} \ge 1$, then $|x_n| \ge 1$, $\forall\, n \ge K$

$\Rightarrow$ $x_n \not\to 0$ as $n \to \infty$

$\Rightarrow$ $\sum x_n$ is divergent ($n^{th}$ Term Test 3.7.3) ∦

**Cor 9.2.3** Suppose $r = \lim_{n\to\infty} |x_n|^{\frac{1}{n}}$ exists.

Then
- $r < 1 \Rightarrow \sum x_n$ is absolutely convergent
- $r > 1 \Rightarrow \sum x_n$ is divergent.

(No conclusion for $r = 1$. see Eg 9.2.7 (b) later)

Pf: If $r<1$, then $\forall\ r<r_1<1$, $\exists\ K\in\mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} \le r_1 < 1, \quad \forall\ n\ge K,$$

then part (a) of Root Test $\Rightarrow$ $\Sigma x_n$ absolutely convergent.

If $r>1$, then $\exists\ K\in\mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} > 1, \quad \forall\ n\ge K,$$

then part (b) of Root Test $\Rightarrow$ $\Sigma x_n$ divergent. #

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**Thm 9.2.4 (Ratio Test) (D'Alembert)**

Let $x_n \ne 0$, $\forall\ n=1,2,3,\cdots$

(a) If $\exists\ 0<r<1$ and $K\in\mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \le r, \quad \forall\ n\ge K,$$

then $\Sigma x_n$ is absolutely convergent

(b) If $\exists\ K\in\mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \ge 1, \quad \forall\ n\ge K,$$

then $\Sigma x_n$ is divergent.

---

Pf: (a) $\forall\ n\ge K$, $|x_n| \le r|x_{n-1}| \le r^2|x_{n-2}| \le \cdots \le r^{n-K}|x_K| \overset{def}{=} y_n$

If $0<r<1$, then $\Sigma y_n = \Sigma r^{n-K}|x_K| = \frac{|x_K|}{r^K}\Sigma r^n$ is convergent

Comparison Test 3.7.7 $\Rightarrow$ $\Sigma |x_n|$ is convergent.

i.e. $\Sigma x_n$ is absolutely convergent.

(b) $\forall n \geq K$, $|x_n| \geq |x_{n-1}| \geq |x_{n-2}| \geq \cdots \geq |x_K|$

$\therefore$ $x_n \not\to 0$ as $n \to \infty$ $\Rightarrow$ $\sum x_n$ is divergent. ✗✗

**Cor 9.2.5** If $\begin{cases} \bullet\ x_n \neq 0,\ \forall n = 1, 2, 3, \cdots, \text{ and} \\ \bullet\ r = \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| \text{ exists} \end{cases}$

Then $\begin{cases} \bullet\ r < 1 \Rightarrow \sum x_n \text{ is } \underline{\text{absolutely convergent}}. \\ \bullet\ r > 1 \Rightarrow \sum x_n \text{ is } \underline{\text{divergent}} \end{cases}$

(No conclusion for $r = 1$. see Eg 9.2.7 (c) later)

Pf: If $r < 1$, then $\forall r_1 \in (r, 1)$, $\exists K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| < r_1 < 1, \quad \forall n \geq K$$

Part (a) of Thm 9.2.4 $\Rightarrow$ $\sum x_n$ is absolutely convergent.

If $r > 1$, then $\exists K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| > 1, \quad \forall n \geq K$$

Part (b) of Thm 9.2.4 $\Rightarrow$ $\sum x_n$ is divergent. ✗✗

# The Integral Test

**Def** (Improper Integral)

For $a \in \mathbb{R}$, if $\begin{cases} \bullet\ f \in R[a,b],\ \forall b > a, \text{ and} \\ \bullet\ \lim_{b \to +\infty} \int_a^b f \text{ exists (and } < +\infty.) \end{cases}$

then the improper integral $\int_a^\infty f$ is defined to be

$$\int_a^\infty f = \lim_{b \to +\infty} \int_a^b f .$$

**Thm 9.2.6** (Integral Test)

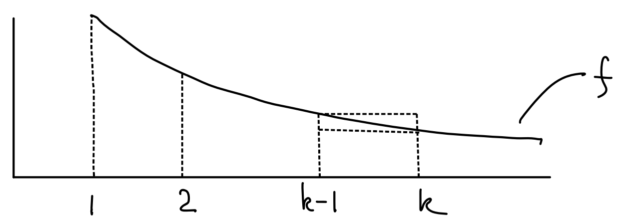
Let $f(t) > 0$, decreasing on $\{t \geq 1\}$.

Then $\sum_{k=1}^{\infty} f(k)$ converges $\Leftrightarrow$ $\int_1^\infty f = \lim_{b \to +\infty} \int_1^b f$ exists.

In this case,

$$\int_{n+1}^{\infty} f(t)dt \leq \sum_{k=1}^{\infty} f(k) - \sum_{k=1}^{n} f(k) \leq \int_{n}^{\infty} f(t)dt, \quad \forall n = 1, 2, \cdots$$

Pf:




$f > 0$ & decreasing $\Rightarrow$ $\forall k = 2, 3, \cdots$

$$f(k) \leq \int_{k-1}^{k} f(t)dt \leq f(k-1) \quad \text{— } (*)_1$$

$$\Rightarrow \sum_{k=2}^{n} f(k) \leq \sum_{k=2}^{n} \int_{k-1}^{k} f(t)dt \leq \sum_{k=2}^{n} f(k-1) \left( = f(1) + \cdots + f(n-1) \right)$$

Let $S_n = \sum_{k=1}^{n} f(k)$

Then, we have

$$S_n - f(1) \leq \int_1^n f(t)\,dt \leq S_{n-1}. \quad (\forall n)$$

$$\left( \Rightarrow \quad \int_1^{n+1} f(t)\,dt \leq S_n \leq f(1) + \int_1^n f(t)\,dt, \quad \forall n \right)$$

$\therefore \lim_{n\to\infty} S_n$ exists $\Leftrightarrow \lim_{n\to\infty} \int_1^n f(t)\,dt$ exists (bdd, increasing)

& $\sum_{k=1}^{\infty} f(k)$ converges $\Leftrightarrow \int_1^{\infty} f$ exists. $\left(\lim_{n\to\infty}\int_1^n f \text{ exists} \Leftrightarrow \lim_{b\to\infty}\int_1^b f \text{ exists}\right)$

see below

Using $(*)_1$ again, if $m > n$, then

$$\sum_{k=n+1}^{m} f(k) \leq \sum_{k=n+1}^{m} \int_{k-1}^{k} f(t)\,dt \leq \sum_{k=n+1}^{m} f(k-1)$$

$$\Rightarrow \quad S_m - S_n \leq \int_n^m f(t)\,dt \leq S_{m-1} - S_{n-1}$$

Hence, $\forall m > n$, we have

$$\int_{n+1}^{m+1} f(t)\,dt \leq S_m - S_n \leq \int_n^m f(t)\,dt$$

Letting $m \to \infty$, we have

$$\int_{n+1}^{\infty} f(t)\,dt < S - S_n \leq \int_n^{\infty} f(t)\,dt$$

where $S = \sum_{k=1}^{\infty} f(k)$.

Finally,

<u>Claim</u>: Let $f > 0$, on $[1, \infty)$; $f \in R[1,b]$, $\forall b > 1$, then

$$\lim_{n\to\infty} \int_1^n f(t)\,dt \text{ exists} \Leftrightarrow \lim_{b\to\infty} \int_1^b f(t)\,dt \text{ exists}.$$

Pf: ($\Rightarrow$) Assume $\lim\limits_{n\to\infty} \int_1^n f$ exists.

$\forall b>1$, $\exists n \in \mathbb{N}$ s.t. $n \leq b < n+1$

(in fact $n$ = largest integer $\leq b$.)

Since $f>0$,

$$\int_1^n f(t)dt \leq \int_1^b f(t)\,dt \leq \int_1^{n+1} f(t)dt$$

Since $b \to \infty \Rightarrow n\to\infty$, and

$$\lim_{n\to\infty}\int_1^n f(t)dt = \lim_{n\to\infty}\int_1^{n+1} f(t)dt$$

$\therefore$ $\lim\limits_{b\to\infty}\int_1^b f(t)\,dt$ exists and $= \lim\limits_{n\to\infty}\int_1^n f(t)dt$

($\Leftarrow$) Assume $\lim\limits_{b\to\infty}\int_1^b f$ exists.

Then subseq $\int_1^n f$ has limit & equals $\lim\limits_{b\to\infty}\int_1^b f$.

This completes the proof of the integral test. #

Egs 9.2.7

(a) Recall Eg 3.7.2(c)

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$ is absolutely convergent.

$$\left(\frac{1}{n(n+1)} > 0\right)$$

Using Limit Comparison Test II (Thm 9.2.1)

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n(n+1)}} = \lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$$

$\Rightarrow$ $\sum \frac{1}{n^2}$ is absolutely convergent.

(b) However, Root Test (Thm 9.2.2) doesn't apply to $\sum \frac{1}{n^2}$

(in fact $\sum \frac{1}{n^p}$, $\forall p > 0$):

$$\begin{cases} \bullet \ \left(\frac{1}{n^p}\right)^{\frac{1}{n}} < 1, \text{ and} \\ \bullet \ \left|\frac{1}{n^p}\right|^{1/n} = \frac{1}{(n^{\frac{1}{n}})^p} \longrightarrow 1 \end{cases}$$ since $n^{\frac{1}{n}} \to 1$

$\therefore$ both conditions in part (a) & part (b) don't hold.

And the Cor 9.2.3 cannot be applied too.

$$\left(r = \lim_{n\to\infty} \left|\frac{1}{n^p}\right|^{\frac{1}{n}} = 1.\right)$$

(c) Ratio Test (Thm 9.2.4) and its Cor 9.2.5 also don't work

for $\sum \frac{1}{n^p}$ :

$$\left| \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} \right| = \frac{n^p}{(n+1)^p} = \frac{1}{\left(1+\frac{1}{n}\right)^p} \to 1$$

$r = 1$,
no information from Ratio test!

(d) (To be cont'd)