Thm 8.2.6 (Dini's Theorem)

Let
- $f_n : [a,b] \to \mathbb{R}$ be a monotone seq. of continuous functions
- $f_n \to f$ on $[a,b]$ (pointwise convergence)
- f is continuous

Then $f_n \rightrightarrows f$ on $[a,b]$ (uniform convergence)

Remark:

monotone $\begin{cases} \text{increasing seq.: } n \le m \Rightarrow f_n(x) \le f_m(x), \ \forall x \in [a,b] \\ \text{decreasing seq.: } n \le m \Rightarrow f_n(x) \ge f_m(x), \ \forall x \in [a,b] \end{cases}$

Pf We assume f_n is a decreasing seq. The proof is similar for increasing sequence.

Let $g_n = f_n - f$.

then $g_n \ge 0$ decreasing, continuous, and

$g_n \to 0$ (pointwise)

(Different proof from the Textbook)

Assume on the contrary that $g_n \not\rightrightarrows 0$ (not uniform).

Then by Lemma 8.1.5,

$\exists \varepsilon_0 > 0$, a subseq g_{n_k} of g_n, and a seq $x_k \in [a,b]$

s.t. $|g_{n_k}(x_k) - 0| \ge \varepsilon_0$

$\Rightarrow \quad g_{n_k}(x_k) \geq \varepsilon_0$

Since $x_k \in [a,b]$, (x_k) is a bounded seq.

Then <u>Bolzano-Weierstrass Thm (Thm 3.4.8)</u> implies that

x_k has a convergence subseq $(x_{k_\ell})_{\ell=1}^{\infty}$

Let $\lim_{\ell\to\infty} x_{k_\ell} = z$.

Since $[a,b]$ is a closed interval, $z \in [a,b]$.

By assumption $g_n(z) \to 0$ as $n \to \infty$.

$\therefore \quad g_{n_{k_\ell}}(z) \to 0$ as $\ell \to \infty$.

$\Rightarrow \quad \exists\, L > 0$ s.t.

if $\ell \geq L$, then $0 \leq g_{n_{k_\ell}}(z) < \frac{\varepsilon_0}{2}$

In particular $0 \leq g_{n_{k_L}}(z) < \frac{\varepsilon_0}{2}$

For clarity of presentation, denote n_{k_L} by N.

Then $$0 \leq g_N(z) < \frac{\varepsilon_0}{2}.$$

Now using continuity of $g_N \ (= g_{n_{k_L}})$

$$\lim_{\ell\to\infty} g_N(x_{k_\ell}) = g_N(z) \qquad \left(\text{since } \lim_{\ell\to\infty} x_{k_\ell} = z\right)$$

$\Rightarrow \quad \exists\, L_1 > 0$ s.t. if $\ell \geq L_1$, then

$$g_N(x_{k_\ell}) < \frac{\varepsilon_0}{2}$$

Using the assumption that g_n is decreasing, we have

$$g_n(x_{k_\ell}) \leq g_N(x_{k_\ell}) < \frac{\varepsilon_0}{2}, \quad \forall n \geq N = n_{k_L}$$

In particular, for $n = n_{k_\ell}$ with $\ell \geq \max\{L, L_1\}$, we have

$$\varepsilon_0 \leq g_{n_{k_\ell}}(x_{k_\ell}) \leq \frac{\varepsilon_0}{2}$$

which is a contradiction.

Therefore $g_n \rightrightarrows 0$ (uniform convergence) $\#$