Def 8.1.7 (Uniform Norm) (supnorm in some other books)

If $\varphi : A \to \mathbb{R}$ is bounded on A (i.e. $\varphi(A)$ is a bounded subset of $\mathbb{R}$), then we define the uniform norm of φ on A by

$$\|\varphi\|_A = \sup\{|\varphi(x)| : x \in A\}.$$

Remark: $\|\varphi\|_A \leq \varepsilon \iff |\varphi(x)| \leq \varepsilon, \forall x \in A$.

Lemma 8.1.8: $f_n \rightrightarrows f$ on A $\iff \|f_n - f\|_A \to 0$.

Pf: ($\Rightarrow$) $f_n \rightrightarrows f$ on A.

By Def 8.1.4, $\forall \varepsilon > 0$, $\exists K(\frac{\varepsilon}{2}) \in \mathbb{N}$

s.t. if $n \geq K(\frac{\varepsilon}{2})$, then

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \forall x \in A$$

$\therefore \forall \varepsilon > 0, \exists N(\varepsilon) = K(\frac{\varepsilon}{2}) \in \mathbb{N}$ s.t. if $n \geq N(\varepsilon)$

$$\|f_n - f\|_A \leq \frac{\varepsilon}{2} < \varepsilon \quad \text{(by remark above)}$$

i.e. $\|f_n - f\|_A \to 0$ as $n \to \infty$.

($\Leftarrow$) If $\|f_n - f\|_A \to 0$. Then $\forall \varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ s.t.

if $n \geq K(\varepsilon)$, $\|f_n - f\|_A < \varepsilon$.

$\Rightarrow |f_n(x) - f(x)| < \varepsilon, \forall x \in A$.

$\therefore f_n \rightrightarrows f$ on A. ※

Eg 8.1.9

(a) eg 8.1.2(a), $f_n(x) = \frac{x}{n}$ on $\mathbb{R}$, $f(x) = 0$, on $\mathbb{R}$.

$f_n(x) - f(x) = \frac{x}{n}$ is unbounded, $\|f_n - f\|_{\mathbb{R}}$ is not defined.

However, if one consider only on the interval $A = [0,1]$.

Then $f_n(x) - f(x) = \frac{x}{n}$ is bounded on $[0,1]$,

and $\|f_n - f\|_{[0,1]} = \sup\{|\frac{x}{n}| : x \in [0,1]\}$

$= \frac{1}{n}$ ($\to 0$ as $n \to \infty$)

$\therefore$ $f_n|_{[0,1]} \rightrightarrows \underset{f}{\underset{\parallel}{0}}$ on $[0,1]$

(in fact $f_n \rightrightarrows f$ on any bounded subset, but $\not\rightrightarrows$ on unbounded subset)