Def 7.3.3: If $f \in \mathcal{R}[a,b]$, then the function defined by

$$F(z) = \int_a^z f \quad \text{for } z \in [a,b],$$

is called the <u>indefinite integral</u> of f with <u>basepoint</u> a.

(One may use other point as base point & is still called indefinite integral (Ex 7.3.6))

Thm 7.3.4 If $f \in \mathcal{R}[a,b]$, then

$$F(z) = \int_a^z f \quad \text{is } \underline{\text{continuous}} \text{ on } [a,b]$$

and in fact, if $|f(x)| \le M$, $\forall x \in [a,b]$, then

$$(*) \qquad |F(z) - F(w)| \le M|z-w|, \quad \forall z, w \in [a,b].$$

<u>Remarks:</u> (i) M exists because $f \in \mathcal{R}[a,b] \Rightarrow f$ is bdd

(ii) $(*)$ is called a <u>Lipschitz condition</u>, much stronger than just continuity.

<u>Pf</u> $\forall z, w \in [a,b]$ with $w \le z$, Additivity Thm 7.2.9 $\Rightarrow$

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f = F(w) + \int_w^z f$$

$$\therefore \quad F(z) - F(w) = \int_w^z f.$$

If $\quad -M \le f(x) \le M, \quad \forall x \in [a,b]$,

Thm 7.1.5 (c) $\Rightarrow \quad -M(z-w) \le \int_w^z f \le M(z-w)$

$\therefore \quad |F(z)-F(w)| = \left|\int_w^z f\right| \le M(z-w) = M|z-w|$

(since $w \le z$)

Clearly, the case $z \le w$ follows immediately too. #

<u>Thm 7.3.5</u> (<u>Fundamental Theorem of Calculus (2nd Form)</u>)

Let $f \in \mathcal{R}[a,b]$ and <u>continuous at c</u>.

Then $F(z) = \int_a^z f$ is <u>differentiable at $z=c$</u> and

$$F'(c) = f(c).$$

Pf We'll prove only for the <u>right-hand derivative</u>

$$\lim_{h \to 0+} \frac{F(c+h)-F(c)}{h} = f(c)$$

The left-hand derivative can be handled similarly.

Therefore, we assume $c \in [a,b)$.

Since f is continuous at c, $\forall \varepsilon > 0$, $\exists \eta_\varepsilon > 0$ s.t. if

(*) $|f(x)-f(c)| < \varepsilon, \quad \forall x \in [c, c+\eta_\varepsilon)$. (consider only right side)

Let $h \in (0, \eta_\varepsilon)$, then Additivity Thm 7.2.8 (Cor 7.2.10)

$\Rightarrow f \in \mathcal{R}[a, c+h]$, $\mathcal{R}[a,c]$ & $\mathcal{R}[c, c+h]$ and

$$\int_a^{c+h} f = \int_a^c f + \int_c^{c+h} f$$

i.e. $\quad F(c+h) - F(c) = \int_c^{c+h} f$

By $(*)$ $\quad f(c) - \varepsilon < f(x) < f(c) + \varepsilon, \ \forall x \in [c, c+\eta_\varepsilon)$

we have $\quad (f(c)-\varepsilon)h \leq \int_c^{c+h} f \leq (f(c)+\varepsilon)h,$

which implies $\quad f(c) - \varepsilon \leq \dfrac{F(c+h)-F(c)}{h} \leq f(c) + \varepsilon$

$$\Rightarrow \quad \left| \frac{F(c+h)-F(c)}{h} - f(c) \right| \leq \varepsilon, \quad \forall h \in (0, \eta_\varepsilon)$$

It proves that $\quad \lim\limits_{h \to 0^+} \dfrac{F(c+h)-F(c)}{h} = f(c)$ $\quad$ ##

<u>Thm 7.3.6</u> If f is <u>continuous</u> on $[a,b]$, then

- $F(x) = \int_a^x f$ is <u>differentiable</u> on $[a,b]$, and
- $F'(x) = f(x)$, $\forall x \in [a,b]$

<u>Pf</u>: f cts on $[a,b] \Rightarrow f \in \mathcal{R}[a,b]$ & cts at every pt. $c \in [a,b]$ ##

Eg 7.3.7

(a) $f(x) = \operatorname{sgn} x$ on $[-1,1]$.

Then • $f \in \mathcal{R}[-1,1]$ (equals a step function except at a point)

• f not continuous at $x=0$, but continuous $\forall x \in [-1,1] \setminus \{0\}$.

Simple calculation: indefinite integral with basepoint -1 is

$$F(x) = \int_{-1}^{x} \operatorname{sgn}(x)dx = |x| - 1 \quad (\text{Ex!})$$

One can see that $F'(0)$ doesn't exist ("f cts at c" is a necessary condition)

and F is not an antiderivative of $f(x) = \operatorname{sgn}(x)$.

(b) Let h = Thomae's function

$$h(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in [0,1] \ \& \ m, n \text{ have no common factors } (\gcd(m,n)=1), \ n \in \mathbb{N} = \{1,2,3,\cdots\}, \ m \neq 0 \\ 1, & \text{if } x = 0 \\ 0, & \text{if } x \text{ is irrational } \& \ x \in [0,1]. \end{cases}$$

Then by Eg 7.1.7, one concludes that

$$H(x) = \int_0^x h \equiv 0, \quad \forall x \in [0,1] \quad (\text{Check!})$$

$\Rightarrow \quad H'(x) = 0$ exists $\forall x \in [0,1]$

However, $H'(x) \neq h(x)$, $\forall$ rational $x \in [0,1]$.

Thm 7.3.8 (Substitution Theorem)

Let
- $f : I \to \mathbb{R}$ cts, (I = interval)
- $\varphi : [\alpha, \beta] \to \mathbb{R}$ s.t. $\varphi'(t)$ exists & cts $\forall t \in [\alpha, \beta]$, (i.e. φ has a continuous derivative)
- $\varphi([\alpha, \beta]) \subset I$

$$\left(\begin{tikzcd} {[\alpha,\beta]} \arrow[r, "\varphi"] \arrow[rr, bend right, "f \circ \varphi"'] & I \arrow[r, "f"] & \mathbb{R} \end{tikzcd} \right)$$

Then $\displaystyle\int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$

(to be cont'd)