**Thm 7.2.8** If $f:[a,b]\to\mathbb{R}$ is monotone on $[a,b]$, $(-\infty<a<b<+\infty)$

then $f\in\mathcal{R}[a,b]$.

Pf: Suppose $f$ is increasing (decreasing are similar)

Take uniform partition $\mathcal{P}=\{[x_{i-1},x_i]\}_{i=1}^{n}$ such that

$$x_i - x_{i-1} = \frac{b-a}{n}, \quad \forall i=1,2,\cdots,n \quad (\text{with } x_0=a).$$

Then $f(x_{i-1}) \le f(x) \le f(x_i)$, $\forall x\in[x_{i-1},x_i]$ $(\forall i=1,\cdots,n)$

Define step functions

$$\alpha_n(x) = \begin{cases} f(x_{i-1}), & x\in[x_{i-1},x_i) \\ f(x_{n-1}), & x\in[x_{n-1},x_n] \end{cases}$$

and

$$\omega_n(x) = \begin{cases} f(x_i), & x\in[x_{i-1},x_i) \\ f(x_n), & x\in[x_{n-1},x_n] \end{cases}$$

Then $\alpha_n(x)\le f(x)\le \omega_n(x)$, $\forall x\in[a,b]$

and

$$\int_a^b \alpha_n = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

$$= \frac{b-a}{n}\left[f(x_0)+f(x_1)+\cdots+f(x_{n-1})\right]$$

$$\int_a^b \omega_n = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

$$= \frac{b-a}{n}\left[f(x_1)+f(x_2)+\cdots+f(x_n)\right]$$

$\therefore \int_a^b (\omega_n - \alpha_n) = \frac{b-a}{n}\left[f(x_n) - f(x_0)\right] = \frac{(b-a)(f(b)-f(a))}{n}$

Hence $\forall \varepsilon > 0$, $\exists n_\varepsilon > \frac{(b-a)(f(b)-f(a))}{\varepsilon}$ s.t.

$$\alpha_{n_\varepsilon}(x) \leq f(x) \leq \omega_{n_\varepsilon}(x), \quad \forall x \in [a,b] \quad \&$$

$$\int_a^b (\omega_{n_\varepsilon} - \alpha_{n_\varepsilon}) < \varepsilon$$

$\therefore f \in \mathcal{R}[a,b]$ by Squeeze Thm 7.2.3 ⋕