Properties of Integral

Thm 7.1.5 Suppose $f, g \in \mathcal{R}[a,b]$. Then

(a) $kf \in \mathcal{R}[a,b]$, $\forall k \in \mathbb{R}$ and

$$\int_a^b kf = k \int_a^b f$$

(b) $f+g \in \mathcal{R}[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

(c) $f(x) \le g(x) \quad \forall x \in [a,b] \Rightarrow \int_a^b f \le \int_a^b g$.

Pf: (a) Ex. (Similar to the proof of (b) & easier)

(b) $f, g \in \mathcal{R}[a,b] \Rightarrow$

$\forall \varepsilon > 0$, $\exists\ \delta_1 > 0$ s.t. $\left| S(f, \dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon$, $\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_1$

& $\exists\ \delta_2 > 0$ s.t. $\left| S(g, \dot{\mathcal{P}}) - \int_a^b g \right| < \varepsilon$, $\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_2$.

Also note that for any $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$,

$$\begin{aligned} S(f+g; \dot{\mathcal{P}}) &= \sum_{i=1}^n (f+g)(t_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) + \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) \\ &= S(f; \dot{\mathcal{P}}) + S(g; \dot{\mathcal{P}}) \end{aligned}$$

Then $\forall\ \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta = \min\{\delta_1, \delta_2\}$ (70), we have

$$\left| S(f+g;\dot{\mathcal{P}}) - \left(\int_a^b f + \int_a^b g\right)\right|$$

$$\leq \left| S(f;\dot{\mathcal{P}}) - \int_a^b f \right| + \left| S(g;\dot{\mathcal{P}}) - \int_a^b g \right|$$

$$< \varepsilon + \varepsilon = 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we've proved that

$$f+g \in \mathcal{R}[a,b] \text{ and } \int_a^b (f+g) = \int_a^b f + \int_a^b g .$$

(c) As in (b), we conclude, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

for $\dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta$,

$$\left| S(f;\dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon \text{ and } \left| S(g;\dot{\mathcal{P}}) - \int_a^b g \right| < \varepsilon$$

$$\Rightarrow \int_a^b f - \varepsilon < S(f;\dot{\mathcal{P}}) \quad \& \quad S(g;\dot{\mathcal{P}}) < \int_a^b g + \varepsilon .$$

Now $f(x) \leq g(x)$, $\forall x \in [a,b] \Rightarrow$

$$S(f;\dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) = S(g;\dot{\mathcal{P}})$$

$$\therefore \int_a^b f - \varepsilon < S(f;\dot{\mathcal{P}}) \leq S(g;\dot{\mathcal{P}}) < \int_a^b g + \varepsilon$$

or $\int_a^b f < \int_a^b g + 2\varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $\int_a^b f \leq \int_a^b g$ $\#$