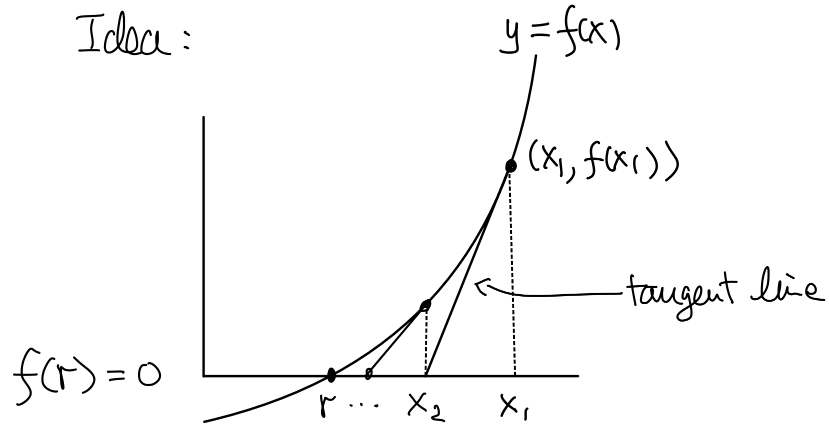
Newton's Method

Goal:

Find root r of f

Idea:



Equation of tangent line: $y - f(x_1) = f'(x_1)(x - x_1)$

$\therefore$ its intersection with x-axis, x_2, satisfies

$$-f(x_1) = f'(x_1)(x_2 - x_1)$$

$\therefore$ $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad (\text{provided } f'(x_1) \neq 0)$$

Successively $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \left(\begin{array}{c}\text{provided } f'(x_k) \neq 0 \\ \text{for } k = 1, \cdots, n\end{array}\right)$$

Hope (to find condition such that)

$$x_n \to r \quad (\text{a zero (root) of } f).$$

Thm 6.4.7 (Newton's Method)

Let
- $f:[a,b]\to\mathbb{R}$ twice differentiable $(a<b)$
- $f(a)f(b)<0$ (ie $f(a)$, $f(b)$ have opposite signs)
- $\exists$ constants $m>0$, $M\geq 0$ such that

$$|f'(x)|\geq m>0 \quad \& \quad |f''(x)|\leq M, \qquad \forall x\in[a,b].$$

Then $\exists$ a subinterval $I^*\subset[a,b]$

- containing a zero r of f such that
- $\forall x_1\in I^*$, the sequence (x_n) defined by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \qquad \forall n=1,2,3\cdots$$

belongs to I^* and

- $\lim_{n\to\infty} x_n = r$

Moreover
- $|x_{n+1}-r|\leq K|x_n-r|^2 \quad \forall n=1,2,3\cdots$

where $K=M/2m$.

Pf: Since $f(a)f(b)<0$, $f(a)$, $f(b)$ have opposite signs (& nonzero)

f twice differentiable $\Rightarrow f$ cts. on $[a,b]$.

Intermediate Thm $\Rightarrow \exists\, r\in(a,b)$ such that $f(r)=0$.

Note that $|f'(x)| \geq m > 0, \ \forall x \in [a,b]$, Rolle's Thm

$\Rightarrow$ r is the unique zero of f in $[a,b]$.

i.e. $f(x) \neq 0, \ \forall x \in [a,b] \setminus \{r\}$, (Ex!)

Now $\forall x' \in I$, Taylor's Thm $\Rightarrow$

$$0 = f(r) = f(x') + f'(x')(r-x') + \frac{f''(c')}{2}(r-x')^2$$

for some c' between r & x'.

(since f is twice diff.)

If $x'' = x' - \dfrac{f(x')}{f'(x')}$, we have

$$x'' = x' + \frac{f'(x')(r-x') + \frac{f''(c')}{2}(r-x')^2}{f'(x')}$$

$$= r + \frac{1}{2}\frac{f''(c')}{f'(x')}(r-x')^2$$

$$\Rightarrow \quad |x''-r| \leq \frac{1}{2}\frac{|f''(c')|}{|f'(x')|}|x'-r|^2$$

$$\leq \frac{1}{2}\frac{M}{m}|x'-r|^2 = K|x'-r|^2. \quad \text{———} \quad (*)$$

Choose $\delta > 0$ such that

$$\delta < 1/K \quad \& \quad [r-\delta, r+\delta] \subset [a,b],$$

and let $I^* = [r-\delta, r+\delta]$

Then, if $x_n \in I^* \ (\subset [a,b])$ for some $n = 1, 2, 3, \cdots$, we have, from (*),

$$|x_{n+1}-r| \le K|x_n-r|^2 \le K\delta^2 < \delta$$

$\therefore$ $x_{n+1} \in I^*$.

i.e. $x_n \in I^* \Rightarrow x_{n+1} \in I^*$.

Therefore, if $x_1 \in I^*$, induction $\Rightarrow$

the sequence $(x_n) \subset I^*$,

and satisfies the required inequality

$$|x_{n+1}-r| \le K|x_n-r|^2, \quad \forall\, n=1,2,3,\dots$$

Finally, to see "limit", we note 1st that

$$|x_{n+1}-r| \le K|x_n-r|^2 \le K\delta\,|x_n-r| \quad \text{———} \quad (*)_2$$

Then iterate $(*)_2$:

$$|x_{n+1}-r| \le (K\delta)|x_n-r| \le (K\delta)\big(K\delta\,|x_{n-1}-r|\big)$$

$$= (K\delta)^2|x_{n-1}-r| \le \dots \le (K\delta)^n|x_1-r|$$

Since $K\delta<1$, $(K\delta)^n \to 0$ as $n\to\infty$,

and $|x_1-r|$ is a constant, we have

$$|x_{n+1}-r| \to 0 \quad \text{as } n\to\infty$$

i.e. $\lim_{n\to\infty} x_n = r$ $\#$

eg 6.4.8 Using Newton's Method to approximate $\sqrt{2}$.

Soln: Convert the problem to a problem of finding root in order to use Newton's Method:

Consider $f(x) = x^2 - 2 \quad \forall x \in \mathbb{R}$.

Calculation: $f'(x) = 2x$ ($\neq 0$ near the root, as 0 is not a root)

(f'' exists and satisfies the condition, but we don't need to find it explicitly in the approximation.)

One need to guess an initial point x_1.

Since $1^2 = 1$, $2^2 = 4$, ($f(1) = -1$, $f(2) = 2$)

it seems reasonable to try $x_1 = 1$.

Note that $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^2 - 2}{2x_n}$

$$= x_n - \frac{1}{2}x_n + \frac{1}{x_n}$$

$$= \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

$\therefore x_1 = 1 \Rightarrow \quad x_2 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} = 1.5$

$x_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{17}{12} \simeq 1.416666$

$\vdots$

(Check!) $\quad x_5 \approx 1.41421356237 2$ (correct to 11 places).

Remarks

(a) (*) can be written as $\left(K|x_{n+1}-r|\right) \le \left(K|x_n-r|\right)^2$

Hence if $K|x_n-r| < 10^{-m}$,

then $K|x_{n+1}-r| < 10^{-2m}$

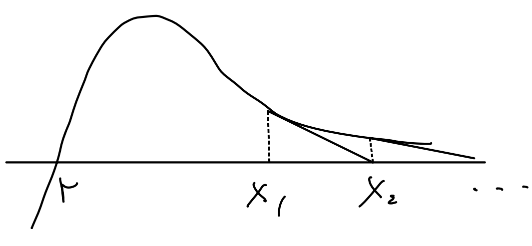
$\therefore$ number of significant digits in $K|x_n-r|$

has been <u>doubled</u>.

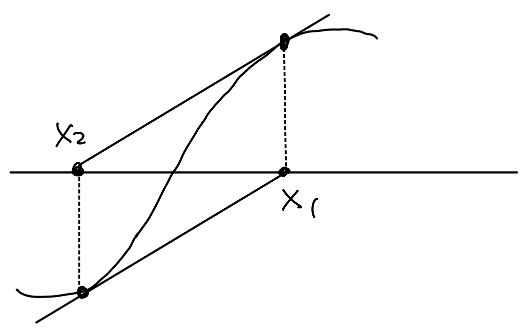
And hence, the sequence (x_n) generated by Newton's method is said to "converge quadratically".

(b) Choose of initial x_1 is important (i.e. has to be in I^*), otherwise (x_n) may not converge to the zero (root).

Possible situations:



$(x_n \to \infty)$



(seq is $(x_1, x_2, x_1, x_2, x_1, x_2, \dots)$)
no limit