(c) Bernoulli's inequality

If $\alpha > 1$, then $(1+x)^\alpha \geq 1 + \alpha x$, $\forall\ x > -1$.

with "equality $\Leftrightarrow$ $x = 0$".

Pf: Consider $h(x) = (1+x)^\alpha$ on $(-1, +\infty)$.

($1+x > 0 \Rightarrow$ taking root of $1+x$ is well-defined for $\alpha \neq$ integer)

Then $h'(x) = \alpha(1+x)^{\alpha-1}$ on $(-1, +\infty)$

(We've proved this in eg 6.1.10(d) for rational α. The case of irrational α will be proved in §8.3)

If $x > 0$, applying MVT to $h(x)$ on $[0, x]$, we have $c \in (0, x)$ such that

$$h(x) - h(0) = h'(c)(x - 0).$$

That is

$$(1+x)^\alpha - 1 = \alpha(1+c)^{\alpha-1} x.$$

Since $c > 0$ & $\alpha - 1 > 0$, we have $(1+c)^{\alpha-1} > 1$.

$\therefore$ $(1+x)^\alpha > 1 + \alpha x$ (The ineq. is strict!)

If $-1 < x < 0$, then applying MVT to $h(x)$ on $[x, 0]$,

we have $c \in (x, 0)$ such that $h(0) - h(x) = h'(c)(0 - x)$

That is

$$1-(1+x)^{\alpha} = \alpha(1+c)^{\alpha-1}(-x)$$

Since $-1<x<c<0$, we have $0<1+c<1$

$$\Rightarrow (1+c)^{\alpha-1} < 1 \qquad (\alpha-1>0)$$

$$\therefore \quad 1-(1+x)^{\alpha} < \alpha(-x) \qquad (\text{as } -x>0)$$

That is $(1+x)^{\alpha} > 1+\alpha x$ (ineq. is strict!)

Clearly $(1+x)^{\alpha} = 1+\alpha x$ for $x=0$.

Therefore $(1+x)^{\alpha} \geq 1+\alpha x$, $\forall x \in (-1,+\infty)$ and

"equality $\Leftrightarrow$ $x=0$". $\#$

(d) If $0<\alpha<1$, then $\forall a>0$ & $b>0$, we have

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b.$$

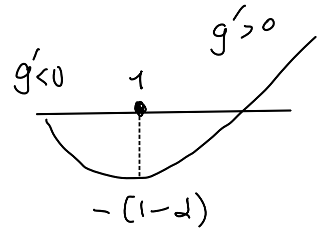
with "equality $\Leftrightarrow$ $a=b$". (Example of application of 1st derivative test)

(Note: for $\alpha=\frac{1}{2}$, we have $\sqrt{ab} \leq \frac{a+b}{2}$)

Pf: Consider $g(x) = \alpha x - x^{\alpha}$ for $x \geq 0$.

Then $g'(x) = \alpha - \alpha x^{\alpha-1} = \alpha(1-x^{-(1-\alpha)})$ $(0<\alpha<1)$

$$\Rightarrow \quad g'(x) \begin{cases} <0 & \text{for } 0<x<1 \\ >0 & \text{for } 1<x \end{cases}$$



Hence $\quad g(x) \geq g(1), \quad \forall x \geq 0 \quad$ and

$$g(x) = g(1) \iff x = 1.$$

That is, $\alpha x - x^{\alpha} \geq \alpha - 1$ or

$$x^{\alpha} \leq \alpha x + (1-\alpha), \quad \forall x \geq 0$$

with "equality $\iff x = 1$".

Now for $a>0, b>0$, put $x = \frac{a}{b} > 0$ into the ineq.,
we have

$$\frac{a^{\alpha}}{b^{\alpha}} \leq \frac{\alpha a}{b} + (1-\alpha)$$

$$\Rightarrow \qquad a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \qquad ⁂$$

Intermediate Value Property of Derivatives (Darboux's Thm)

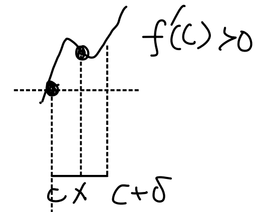
Lemma 6.2.11 Let

- I be an interval and $c \in I$.
- $f: I \to \mathbb{R}$ and $f'(c)$ exists.

Then

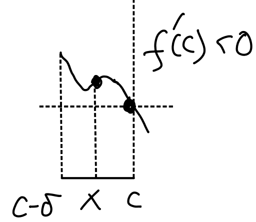
(a) If $f'(c) > 0$, then $\exists\, \delta > 0$ s.t.

$$f(x) > f(c) \quad \forall\, x \in (c, c+\delta) \cap I$$



(b) If $f'(c) < 0$, then $\exists\, \delta > 0$ s.t.

$$f(x) > f(c) \quad \forall\, x \in (c-\delta, c) \cap I$$



Pf: (a) Since $\displaystyle\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) > 0$, (Thm 4.2.9 of the textbook, MATH2050)

$$\exists\, \delta > 0 \text{ s.t. } \frac{f(x) - f(c)}{x - c} > 0, \quad \forall\, x \in (c-\delta, c+\delta) \cap I$$

$$\therefore \quad f(x) - f(c) > 0, \quad \forall\, x \in (c, c+\delta) \cap I.$$

(b) Similarly

$$\exists\, \delta > 0 \text{ s.t. } \frac{f(x) - f(c)}{x - c} < 0, \quad \forall\, x \in (c-\delta, c+\delta) \cap I$$

$$\therefore \quad f(x) - f(c) > 0, \quad \forall\, x \in (c-\delta, c) \cap I.$$

#

Thm 6.2.12 (Darboux's Thm)

If
- f is differentiable on $[a,b]$
- k is a number between $f'(a)$ and $f'(b)$, $(f'(a)\neq f'(b))$

then $\exists\ c\in(a,b)$ such that

$$f'(c)=k.$$

(Remark: No continuity of f' is assumed. Hence the usual Intermediate Value Thm of continuous function doesn't apply.)

Pf: Suppose $f'(a)<f'(b)$ and $f'(a)<k<f'(b)$.

Define $g(x)=kx-f(x)$, $\forall x\in[a,b]$.

Then f differentiable $\Rightarrow$

g is differentiable & hence continuous on $[a,b]$

In particular, g attains a maximum value on $[a,b]$.

Note that $g'(a)=k-f'(a)>0$.

By Lemma 6.2.11, $\exists\ \delta>0$ s.t.

$$g(x)>g(a),\quad \forall x\in(a,a+\delta)\cap[a,b].$$

$\therefore$ a is not the maximum of g

Also $g'(b) = k - f'(b) < 0$, lemma 6.2.11 implies

$\exists \delta > 0$ s.t. $g(x) > g(b)$, $\forall x \in (b-\delta, b) \cap [a,b]$.

$\therefore$ b is not the maximum of g.

Together $\Rightarrow$ g attains its maximum at an interior point $c \in (a,b)$.

Then Interior Extremum Thm (Thm 6.2.1) implies

$$0 = g'(c) = k - f'(c).$$

If $f'(b) < f'(a)$, consider $(-f)$ and we can find similarly a $c \in (a,b)$ s.t. $f'(c) = k$. $\#$

Eg 6.2.13 The signum function $g(x) = \mathrm{sgn}(x)$ restricted on $[-1,1]$:

$$g(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0, & x = 0 \\ -1, & -1 \le x < 0 \end{cases}$$

doesn't satisfy the intermediate value property,

$(1 = g(1), -1 = g(-1),$ & $-1 < \frac{1}{2} < 1$, but no $x \in (-1,1)$ s.t. $g(x) = \frac{1}{2})$

Therefore $g(x) \neq f'(x)$ for any differentiable function f on $[-1,1]$.

(i.e. The differential eqt $\frac{df}{dx} = g$ has no solution on $[-1,1]$)