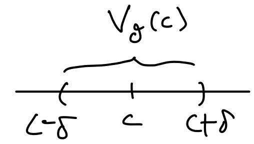
§6.2 The Mean Value Theorem

<u>Recall</u>: function $f: I \to \mathbb{R}$ is said to have a

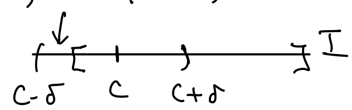


- <u>relative maximum</u> at $c \in I$

  if $\exists$ a neighborhood of c, $V = V_\delta(c) = (c-\delta, c+\delta)$ $(\delta > 0)$, such that

  $$f(x) \le f(c), \quad \forall x \in V \cap I;$$

  (in general, some part may be out of I

)

- <u>relative minimum</u> at $c \in I$

  if $\exists$ a neighborhood of c, $V = V_\delta(c) = (c-\delta, c+\delta)$ $(\delta > 0)$, such that

  $$f(x) \ge f(c), \quad \forall x \in V \cap I;$$

- <u>relative extremum</u> at $c \in I$ if either "relative maximum"

  or "relative minimum"

<u>Thm 6.2.1</u> (<u>Interior Extremum Theorem</u>) (Same notations as above)

Let
- c be an <u>interior</u> point of the interval I
- f has a <u>relative extremum</u> at c.

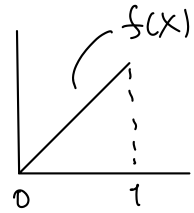
If $f'(c)$ exists, then $f'(c) = 0$.

<u>Note</u>: The condition that $c \in I$ is an interior point is <u>neccessary</u>:

eg: $f(x)=x$ on $[0,1]$ has relative extremum

at $x=0$ (min), but $f'(0)=1\neq 0$.

(at $x=1$ (max), but $f'(1)=1\neq 0$.)



Pf (of Thm 6.2.1):

Prove only the case of relative maximum. The case of relative minimum is similar.

Let $c\in$ interior of I, f has a relative maximum at c and $f'(c)$ exists.

Suppose on the contrary that $f'(c)\neq 0$, then either

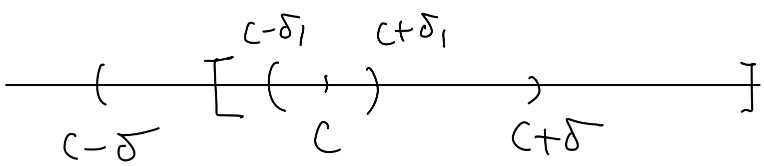
$$f'(c)>0 \quad \text{or} \quad f'(c)<0.$$

If $f'(c)>0$, i.e. $\displaystyle\lim_{\substack{x\to c\\(x\neq c)}} \frac{f(x)-f(c)}{x-c}>0$.

Then (by Thm 4.2.9 of the Textbook, MATH2050), $\exists$ a nbd. $V=V_\delta(c)$

such that $\displaystyle\frac{f(x)-f(c)}{x-c}>0 \quad \forall\ x\in V\cap I,\ x\neq c.$

Since $c\in$ interior of I, one can find a δ_1, $0<\delta_1<\delta$ (if needed) so that $(c-\delta_1, c+\delta_1)\subset V\cap I$.



Note that f has a relative maximum, there exists $\delta_2 > 0$ such that $f(x) \leq f(c)$, $\forall\, x \in (c-\delta_2, c+\delta_2) \cap I$

Then for $\delta_3 = \min\{\delta_1, \delta_2\} > 0$,

$$(c-\delta_3, c+\delta_3) \subset V \cap I \text{ and}$$

$$(c-\delta_3, c+\delta_3) \subset (c-\delta_1, c+\delta_1) \cap I \ \& \ (c-\delta_2, c+\delta_2) \cap I$$

As a result,

$$\left.\begin{aligned} &\frac{f(x)-f(c)}{x-c} > 0, \\ &\text{and } f(x) \leq f(c) \end{aligned}\right\} \quad \forall\, x \in (c-\delta_3, c+\delta_3),\ x \neq c.$$

Since $(c, c+\delta_3) \subset (c-\delta_3, c+\delta_3) \subset V \cap I$

The 1st inequality implies

$\exists\, x > c$, in $(c-\delta_3, c+\delta_3)$ s.t.

$$\frac{f(x)-f(c)}{x-c} > 0 \Rightarrow f(x) - f(c) > 0,$$

which contradicts the 2nd inequality.

Similarly, if $f'(c) < 0$, one can find $\delta_3' > 0$ so that

$$\left.\begin{aligned} &\frac{f(x)-f(c)}{x-c} < 0, \\ &\text{and } f(x) \leq f(c) \end{aligned}\right\} \quad \forall\, x \in (c-\delta_3', c+\delta_3'),\ x \neq c.$$

The 1st inequality $\Rightarrow \exists\, x < c$ such that $\dfrac{f(x)-f(c)}{x-c} < 0$.

$\Rightarrow \quad f(x) - f(c) > 0$ contradicts the 2^{nd} inequality.

All together, we have

$$f'(c) = 0 . \quad \#$$

Cor 6.2.2 Let
- $f : I \to \mathbb{R}$ continuous
- f has a relative extremum at an interior point $c \in I$.

Then either
- $f'(c)$ doesn't exist

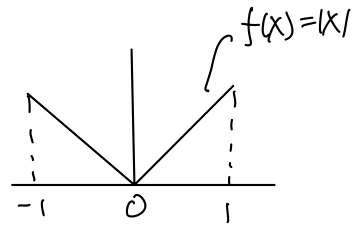
or
- $f'(c) = 0$.

(Pf: Follow easily from Thm 6.2.1)

eg: $f(x) = |x|$ on $I = [-1, 1]$.

interior minimum at $x = 0$.

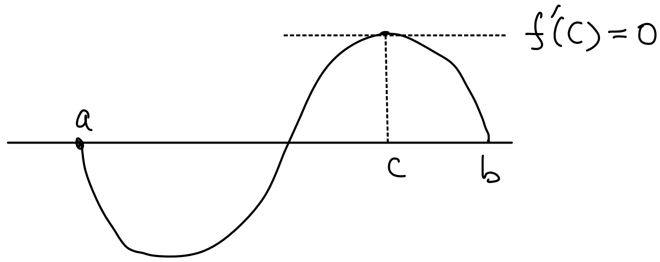
$f'(0)$ doesn't exist



Thm 6.2.3 (**Rolle's Theorem**)

Suppose $(a<b)$
- $f : [a,b] \to \mathbb{R}$ continuous (on closed interval $I = [a,b]$)
- $f'(x)$ exists $\forall\, x \in (a,b)$ (open interval, interior of I)
- $f(a) = f(b) = 0$

Then $\exists\ c \in (a,b)$ such that $f'(c) = 0$



Pf : If $f(x) \equiv 0$ on $[a,b]$, then $f'(x) = 0\ \forall\, x \in [a,b]$. We're done.

If $f(x) \not\equiv 0$, then either $f > 0$ for some point in (a,b)

or $f < 0$ for some point in (a,b).

Note that f is continuous on the closed interval $[a,b]$,

f attains an absolute maximum and an absolute minimum on I.

(Thm 5.3.4 of the Textbook, MATH 2050)

Hence, if $f > 0$ for some point in (a,b), f attains

the absolute maximum, ie. the value $\sup\{f(x) : x \in I\} > 0$,

at some point $c \in (a,b)$ as $f(a) = f(b) = 0$.

Since $c \in (a,b)$, $f'(c)$ exists.

By Interior Extreme Theorem (Thm 6.2.1), $f'(c)=0$.

If there is no $x \in (a,b)$ s.t. $f>0$, then we must have

$f<0$ for some $x \in (a,b)$. Hence $(-f)>0$ for some $x \in (a,b)$

and $-f$ satisfies all conditions as f. Therefore,

$\exists\, c \in (a,b)$ such that $(-f)'(c)=0 \Rightarrow f'(c)=0$. #

<u>Thm 6.2.4</u> (<u>Mean Value Theorem</u>)

Suppose • $f:[a,b] \to \mathbb{R}$ continuous $(a<b)$

• $f'(x)$ exists $\forall\, x \in (a,b)$

Then $\exists$ a point $c \in (a,b)$ such that

$$f(b)-f(a)=f'(c)(b-a)$$

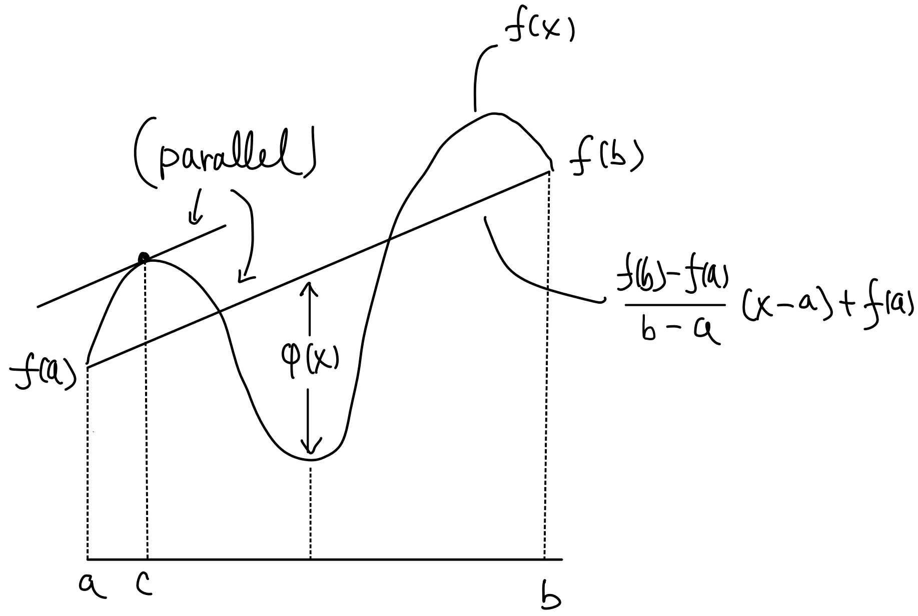
<u>Pf</u>: Consider the function defined on $[a,b]$:

$$\varphi(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a)+f(a)\right]$$

$$= f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

Then φ is continuous on $[a,b]$ as f is continuous on $[a,b]$,

and $\varphi'(x)$ exists $\forall\, x \in (a,b)$ as $f'(x)$ exists $\forall\, x \in (a,b)$.



At the end points

$$\varphi(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = 0$$

$$\varphi(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) = 0$$

$\therefore$ φ satisfies all conditions in Rolle's Thm (Thm 6.2.3).

Hence $\exists\ c \in (a,b)$ such that

$$0 = \varphi'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} \qquad (\text{by Thm 6.1.3 and } (x)' = 1)$$

$$\therefore\ f(b) - f(a) = f'(c)(b-a).$$ ※

Applications of Mean Value Theorem

> **Thm 6.2.5** Suppose • $f:[a,b]\to\mathbb{R}$ continuous $(a<b)$
>
> • $f'(x)$ exists $\forall x\in(a,b)$ (ie. f differentiable on (a,b))
>
> • $f'(x)=0$, $\forall x\in(a,b)$.
>
> Then f is a constant on $[a,b]$.

Pf: let $x\in[a,b]$ and $x>a$.

Applying Mean Value Thm to $f:[a,x]\longrightarrow\mathbb{R}$,

(which clearly satisfies all conditions of the Thm)

we find a point $c\in(a,x)$ such that

$$f(x)-f(a)=f'(c)(x-a)=0 \quad \text{(by assumption } f'(c)=0)$$

$\Rightarrow$ $f(x)\equiv f(a)$, $\forall x\in I$.

$\therefore$ f is constant on I. ⋕

> **Cor 6.2.6** Suppose • $f,g:[a,b]\longrightarrow\mathbb{R}$ continuous
>
> • f,g differentiable on (a,b)
>
> • $f'(x)=g'(x)$, $\forall x\in(a,b)$.
>
> Then $\exists$ constant C such that $f=g+C$ on $[a,b]$.

<u>Recall</u> $f: I \to \mathbb{R}$ is said to be

- <u>increasing</u> on I if $x_1 < x_2$ $(x_1, x_2 \in I) \Rightarrow f(x_1) \leq f(x_2)$

note: "not $<$"

- <u>decreasing</u> on I if $-f$ is increasing on I.

<u>Thm 6.2.7</u> Let $f: I \to \mathbb{R}$ be <u>differentiable</u>. Then

(a) f is increasing on $I \Leftrightarrow f'(x) \geq 0, \ \forall x \in I$

(b) f is decreasing on $I \Leftrightarrow f'(x) \leq 0, \ \forall x \in I$

<u>Pf</u>: (a) $(\Leftarrow)$ Let $f'(x) \geq 0, \ \forall x \in I$.

Then for any $x_1, x_2 \in I$ with $x_1 < x_2$, we can apply the Mean Value Thm to $f: [x_1, x_2] \to \mathbb{R}$

(since f is differentiable on $I \Rightarrow f: [x_1, x_2] \to \mathbb{R}$ satisfies all conditions of MVT)

and find a point $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

$$\geq 0 \quad \text{since } f'(c) \geq 0 \ \& \ x_2 > x_1.$$

$\therefore$ f is increasing on I.

(a) $(\Rightarrow)$ Suppose f is differentiable and increasing on I.

Then $\forall c \in I$, we have

$$\frac{f(x) - f(c)}{x - c} \geq 0, \quad \forall x \in I, \ x \neq c$$

because "f is increasing" $\left(\begin{array}{l} \text{both "positive (or zero)" if } x>c, \\ \text{both "negative (or zero)" if } x<c \end{array} \right)$

Hence f differentiable at c $\Rightarrow$

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \geq 0$$

(b) Applying (a) to $-f$.

<u>Remarks</u>:

(1) <u>Strictly increasing</u> : $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

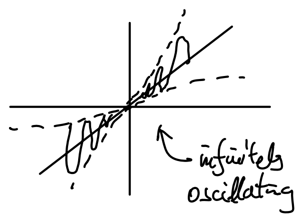
Then ex.13 of §6.2 $\Rightarrow$ "$f'(x) > 0$ on $I \Rightarrow f$ is strictly increasing on I".

<u>But</u> : "$f'(x)>0$ on $I \;\cancel{\Leftarrow}\; f$ is strictly increasing on I".

<u>Counterexample</u> : $f(x) = x^3 : \mathbb{R} \to \mathbb{R}$ is strictly increasing,

but $f'(0) = 0$ which is <u>not</u> ">0".

(2) Consider $g(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.



<u>Exercise 10 of §6.2</u> : $g'(0) = 1 > 0$, but $g(x)$ is <u>not</u> increasing in any neighborhood of 0.

(That is, $g'(x) > 0$ <u>only at a point x</u> is not sufficient to ensure $g(x)$ is increasing near the point. We need $g' > 0$ on an interval!)

Thm 6.2.8 (First Derivative Test for Extrema)

Let
- $f : [a,b] \to \mathbb{R}$ continuous $(a<b)$
- $c \in (a,b)$
- f is differentiable on (a,c) and (c,b).

Then (a) If $\exists\, \delta > 0$ s.t.
$$\begin{cases} \bullet\ (c-\delta, c+\delta) \subseteq [a,b] \\ \bullet\ f'(x) \geq 0 \text{ for } x \in (c-\delta, c) \\ \bullet\ f'(x) \leq 0 \text{ for } x \in (c, c+\delta) \end{cases}$$
then f has a relative maximum at c.

(b) If $\exists\, \delta > 0$ s.t.
$$\begin{cases} \bullet\ (c-\delta, c+\delta) \subseteq [a,b] \\ \bullet\ f'(x) \leq 0 \text{ for } x \in (c-\delta, c) \\ \bullet\ f'(x) \geq 0 \text{ for } x \in (c, c+\delta) \end{cases}$$
then f has a relative minimum at c.

Pf: (a) If $x \in (c-\delta, c)$, then Mean Value Thm

(applying to $f : [x, c] \to \mathbb{R}$) implies $\exists\, c_x \in (x, c)$ s.t.

$$f(c) - f(x) = f'(c_x)(c - x)$$
$$\geq 0 \quad (\text{since } f' \geq 0 \text{ on } (c-\delta, c))$$

If $x \in (c, c+\delta)$, then Mean Value Thm

(applying to $f: [c,x] \to \mathbb{R}$) implies $\exists\, c_x \in (c,x)$ s.t.

$$f(x) - f(c) = f'(c_x)(x-c)$$

$$\leq 0 \quad (\text{since } f' \leq 0 \text{ on } (c, c+\delta))$$

Together we have $f(c) \geq f(x) \quad \forall\, x \in (c-\delta, c+\delta)$

$\therefore$ f has a relative maximum at c.

(b) Applying (a) to $-f$. ※

Remark: Converse of Thm 6.2.8 is not true.

i.e. $\exists$ differentiable function f has a relative maximum at c,

but the statement

"$\exists\, \delta > 0$ s.t. $\begin{cases} \bullet\ (c-\delta, c+\delta) \subseteq [a,b] \\ \bullet\ f'(x) \geq 0 \text{ for } x \in (c-\delta, c) \\ \bullet\ f'(x) \leq 0 \text{ for } x \in (c, c+\delta) \end{cases}$"

is not true (Exercise 9 of §6.2)

Further Applications of the Mean Value Theorem

Examples 6.2.9

(a) Rolle's Thm 6.2.3 can be used to "locate" roots of a function.

In fact, Rollé's Thm $\Rightarrow$

$g = f'$ always has a root between any two zeros of f

(provided f is differentiable & etc.)

explicit eg: $\begin{cases} g(x) = \cos x = (\sin x)' \\ \sin x = 0 \text{ for } x = n\pi \text{ for } n \in \mathbb{Z}. \end{cases}$

Rolle's $\Rightarrow$ $\cos x$ has a root in $(n\pi, (n+1)\pi)$, $\forall n \in \mathbb{Z}$.

(eg. of Bessel functions J_n is omitted)

(b) Using Mean Value Theorem for approximate calculations & error estimates.

eg. Approximate $\sqrt{105}$.

Applying Mean Value Thm to $f(x) = \sqrt{x}$ on $[\overset{a}{\overset{\shortparallel}{100}}, \overset{b}{\overset{\shortparallel}{105}}]$,

$f(105) - f(100) = f'(c)(105 - 100)$ for some $c \in (100, 105)$.

In eg 6.1.10 (d), we've seen that $f'(c) = \dfrac{1}{2\sqrt{c}}$.

$\therefore$ $\sqrt{105} - \sqrt{100} = \dfrac{5}{2\sqrt{c}}$ for some $c \in (100, 105)$

$$\Rightarrow \quad 10 + \frac{5}{2\sqrt{105}} < \sqrt{105} < 10 + \frac{5}{2\sqrt{100}} = 10 + \frac{5}{2\cdot 10} = 10.25$$

And $\sqrt{105} < \sqrt{121} = 11 \Rightarrow \sqrt{105} > 10 + \frac{5}{2\cdot 11}$

Hence $$\frac{205}{22} < \sqrt{105} < \frac{41}{4}$$

(Of course, the estimate can be improved by more careful analysis)

<u>Examples 6.2.10</u> (<u>Inequalities</u>)

(a) $e^x \geq 1+x$, $\forall\, x \in \mathbb{R}$ and "equality $\Leftrightarrow x=0$".

<u>Pf</u>: We will use the fact that

$f(x) = e^x$ has derivative $f'(x) = e^x$, $\forall x \in \mathbb{R}$

(and $f'(0)=1$)

and $e^x > 1$ for $x > 0$

$e^x < 1$ for $x < 0$.

(To be defined and proved in §8.3.)

If $x=0$, then $e^x = 1 = 1+x$. We're done.

If $x>0$, applying MVT (Mean Value Thm) to

$f(x) = e^x$ on $[0, x]$,

we have $c \in (0, x)$ such that

$$e^x - e^0 = e^c (x-0)$$

$$\therefore \quad e^x - 1 > x .$$

If $x<0$, applying MVT to $f(x)=e^x$ on $[x, 0]$,

we have $c \in (x, 0)$ such that

$$e^0 - e^x = e^c (0-x)$$

$$1 - e^x < -x \qquad (e^c < 1 ,\ -x > 0)$$

$$\therefore \quad e^x > 1+x \ , \ \forall\, x<0 .$$

Finally, one observes, in both cases, the inequality is strict.

So "equality $\Leftrightarrow x=0$". $\#$

(b) $-x \le \sin x \le x \ , \ \forall\, x \ge 0$.

Pf: The inequalities are clear for $x=0$.

Let $x>0$. Consider $g(x)=\sin x$ on $[0, x]$.

Then MVT implies $\exists\, c \in (0, x)$ s.t.

$$\sin x - \sin 0 = (\cos c)(x-0)$$

Using $-1 \le \cos c \le 1$ and $\sin 0 = 0$, we have

$$-x \le \sin x \le x \qquad (\text{as } x>0) \quad \#$$