Ch 6 Differentiation

§6.1 The Derivative

Def 6.1.1 • Let • $I \subseteq \mathbb{R}$ be an interval
• $f : I \to \mathbb{R}$ a function on I
• $c \in I$.

We say that $L \in \mathbb{R}$ is the derivative of f at c if $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon, \quad \forall x \in I \text{ with } 0 < |x - c| < \delta(\varepsilon).$$

• In this case we say that f is differentiable at c, and we write $f'(c)$ for L.

Remark: • If limit exists, $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

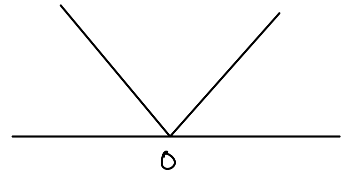
• c may be the endpoint of I (if I is "closed" at c)

then $\lim_{x \to c}$ means $\lim_{\substack{x \to c \\ x \in I}}$ (one sided limit)

• f' defines a function whose domain is a subset of I.

eg $f : (-\infty, \infty) \to \mathbb{R}$

$x \mapsto f(x) = |x|$



Then $f' : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$ given by

$$f'(x) = \begin{cases} 1, & x \in (0, \infty) \\ -1, & x \in (-\infty, 0) \end{cases}$$ and

$f'(0)$ doesn't exist (i.e. $|x|$ is not differentiable at $x = 0$)

Pf: For $c > 0$, then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{|x| - |c|}{x - c} = \lim_{x \to c} \frac{x - c}{x - c}$$

$$= \lim_{x \to c} 1 = 1 \qquad \left(\begin{array}{c} \text{as } x > 0 \\ \text{near } c > 0 \end{array}\right)$$

For $c < 0$, then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{|x| - |c|}{x - c} = \lim_{x \to c} \frac{-x + c}{x - c}$$

$$= \lim_{x \to c} -1 = -1 \qquad \left(\begin{array}{c} \text{as } x < 0 \\ \text{near } c < 0 \end{array}\right)$$

For $c = 0$, then

$$\lim_{\substack{x \to 0 \\ (x \neq 0)}} \frac{f(x) - f(c)}{x - c} = \lim_{x \to 0} \frac{|x|}{x} \text{ doesn't exist}$$

since the two one-sided limits are not equal:

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1 \neq 1 = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} \frac{|x|}{x}$$

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Note: The same argument shows that for $f(x)=x$, $x\in\mathbb{R}$,

f is differentiable $\forall x\in\mathbb{R}$ and

$$f'(x)=1, \quad \forall x\in\mathbb{R}.$$

Thm 6.1.2 (Same notations as in Def 6.1.1)

If $f: I\to\mathbb{R}$ has a derivative at $c\in I$ (ie. differentiable at c), then f is continuous at c.

Pf: For $x\in I$ & $x\neq c$, we have

$$f(x)-f(c)=\frac{f(x)-f(c)}{x-c}\cdot(x-c)$$

$f'(c)$ exists $\Rightarrow$ $$\lim_{\substack{x\to c\\(x\neq c)}}\big(f(x)-f(c)\big)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot\lim_{x\to c}(x-c)$$

$$=f'(c)\cdot 0=0$$

Hence $\lim_{x\to c}f(x)=f(c)$ $\therefore$ f is continuous at c. #

Remarks: • Previous eg $f(x)=|x|$ clearly shows that the converse of Thm 6.1.2 is not true (i.e. continuous at c $\not\Rightarrow$ differentiable at c)

• In fact, there exist continuous but nowhere differentiable functions (will be proved in MATH3060.)

Thm 6.1.3 (Same notations as in Def. 6.1.1)

Let $f: I \to \mathbb{R}$ & $g: I \to \mathbb{R}$ be functions that are differentiable at $c \in I$. Then

(a) If $\alpha \in \mathbb{R}$, the function αf is also differentiable at c, and

$$(\alpha f)'(c) = \alpha f'(c)$$

(b) The function $f+g$ is differentiable at c, and

$$(f+g)'(c) = f'(c) + g'(c)$$

(c) (Product Rule) The function fg is differentiable at c, and

$$(fg)'(c) = f'(c)\,g(c) + f(c)\,g'(c)$$

(d) (Quotient Rule) If $g(c) \neq 0$, then the function f/g is differentiable at c, and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)\,g(c) - f(c)\,g'(c)}{(g(c))^2}$$

(Pfs are easy, just using suitable algebraic expressions and taking limits. We prove only the Quotient Rule here as example, you should do others by yourself.)

Pf of (d):

- Thm 6.1.2 implies that g is continuous at c (as g is diff. at c)
- Then $g(c) \neq 0 \Rightarrow$ there exists an interval $J \subseteq I$ with

  $c \in J$ such that $g(x) \neq 0, \forall x \in J$.

  (Thm 4.2.9 of the text book, MATH2050)

- $q = \frac{f}{g}$ is well-defined function on J and

  $\forall x \in J, \quad x \neq c$, we have

$$\frac{q(x)-q(c)}{x-c} = \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x-c} = \frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)(x-c)}$$

$$= \frac{(f(x)-f(c))g(c) - f(c)(g(x)-g(c))}{g(x)g(c)(x-c)}$$

$$= \frac{1}{g(x)g(c)} \cdot \left[\frac{f(x)-f(c)}{x-c} \cdot g(c) - f(c) \cdot \frac{g(x)-g(c)}{x-c} \right]$$

f, g differentiable at $c \Rightarrow \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c)$

$$\lim_{x \to c} \frac{g(x)-g(c)}{x-c} = g'(c)$$

$$\lim_{x \to c} g(x) = g(c) (\neq 0)$$

$\therefore \lim_{x \to c} \frac{q(x)-q(c)}{x-c}$ exists and

$$q'(c) = \frac{1}{g(c)^2} \left[f'(c)g(c) - f(c)g'(c) \right]$$

$\divideontimes$

$\underline{\text{Cor 6.1.4}}$ If $f_1, \cdots, f_n$ are functions on an interval I to $\mathbb{R}$ that are differentiable at $c \in I$, then

(a) The function $f_1 + \cdots + f_n$ is differentiable at c, and

$$(f_1 + \cdots + f_n)'(c) = f_1'(c) + \cdots + f_n'(c)$$

(b) The function $f_1 \cdots f_n$ is differentiable at c, and

$$(f_1 \cdots f_n)'(c) = f_1'(c) f_2(c) \cdots f_n(c) + f_1(c) f_2'(c) \cdots f_n(c)$$
$$+ \cdots + f_1(c) f_2(c) \cdots f_n'(c)$$

Pf: Just by induction using Thm 6.1.3. ※

Remark: Quotient rule (Thm 6.1.3(d)) together with (b) in Cor 6.1.4

$$\Rightarrow \boxed{(x^n)' = n x^{n-1}, \ \forall n \in \mathbb{Z}} \quad (\forall x \neq 0 \text{ if } n < 0)$$

Pf: Applying (b) in Cor 6.1.4 to the case that

$$f_1 = \cdots = f_n = f \text{ (differentiable)},$$

then for $n \geq 1$, $$(f^n)' = (\underbrace{f \cdots f}_{n})' = \underbrace{f' \underbrace{f \cdots f}_{n-1} + f f' \cdots f + \cdots + \underbrace{f \cdots f}_{n-1} \cdot f'}_{n}$$

$$= n f^{n-1} f'.$$

We've proved that $(x)' = 1$, and hence

$$(x^n)' = n \cdot x^{n-1} \cdot 1 = n x^{n-1}$$

If $n=0$, then $f(x)=x^0 \equiv 1 \Rightarrow f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=0, \forall c$

$$\therefore (x^0)' \equiv 0 \equiv 0\cdot x^{-1}$$

(Note: strictly speaking, the RHS is not defined at $x=0$, but we may interpret the expression nx^{n-1} for $n=0$ as the continuous extension to the zero function on the whole $\mathbb{R}$)

If $n=-m<0$ $(m>0)$, then for $x\neq 0$,

$$(x^n)' = \left(\frac{1}{x^m}\right)' = -\frac{(x^m)'}{(x^m)^2} \quad \text{by Quotient rule}$$

$$= -\frac{mx^{m-1}}{(x^m)^2} = (-m)\cdot x^{(-m)-1} = nx^{n-1} \quad (\text{for } x\neq 0)$$

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Chain Rule

<u>Thm 6.1.5</u> (<u>Carathéodory's Thm</u>) (Same notations as in Def 6.1.1)

f is differentiable at c

$\Leftrightarrow \exists\ \varphi: I \to \mathbb{R}$ <u>continuous at c</u> such that

$$f(x)-f(c)=\varphi(x)(x-c), \quad \forall x\in I.$$

In this case, $\varphi(c)=f'(c)$

Pf: $(\Rightarrow)$ If $f'(c)$ exists, define $\varphi: I\to\mathbb{R}$ by

$$\varphi(x)=\begin{cases}\dfrac{f(x)-f(c)}{x-c}, & x\neq c,\ x\in I\\ f'(c), & x=c.\end{cases}$$

Then $\lim_{\substack{x \to c \\ (x \neq c)}} (\varphi(x) - \varphi(c)) = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} - f'(c) \right) = 0$

$\therefore$ φ is continuous at c.

And clearly $f(x) - f(c) = \varphi(x)(x - c)$ for $x \neq c$,

which is also true trivially at $x = c$ since both sides equal zero.

($\Leftarrow$) If $\exists\, \varphi : I \to \mathbb{R}$ continuous at c such that

$$f(x) - f(c) = \varphi(x)(x - c), \quad \forall x \in I.$$

Then for $x \neq c$, $\dfrac{f(x) - f(c)}{x - c} = \varphi(x) \to \varphi(c)$ as $x \to c$

$\therefore$ $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists & $= \varphi(c)$.

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eg: $f(x) = x^3 : (-\infty, \infty) \to \mathbb{R}$

Then $f(x) - f(c) = x^3 - c^3 = (x^2 + cx + c^2)(x - c)$

$$= \varphi(x)(x - c)$$

where $\varphi(x) = x^2 + cx + c^2$ is continuous at c and

$$\varphi(c) = 3c^2 = f'(c).$$

Thm 6.1.6 (Chain Rule)

$$c \in J \xrightarrow{f} I \ni f(c) \xrightarrow{g} \mathbb{R}$$

Let
- I, J be intervals in $\mathbb{R}$,
- $g : I \to \mathbb{R}$
- $f : J \to \mathbb{R}$ with $f(J) \subseteq I$ (may just assume $f : J \to I$)
- $c \in J$.

If f is differentiable at c and g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and

$$(g\circ f)'(c) = g'(f(c))\, f'(c).$$

Other notations for f': Df or $\frac{df}{dx}$ (when x is the indep. variable)

The formula can be written as $(g\circ f)' = (g'\circ f)\cdot f'$ or

$$\boxed{D(g\circ f) = (Dg \circ f)\cdot Df}$$

Pf: Since $f'(c)$ exists, Carathéodory's Thm 6.1.5

$\Rightarrow \exists\ \varphi : J \to \mathbb{R}$ continuous at c such that

$$f(x) - f(c) = \varphi(x)(x-c), \quad \forall\, x \in J$$

and $\varphi(c) = f'(c)$.

Denote $f(c) = d$, then similarly, $g'(d)$ exists

$\Rightarrow \exists\ \psi : I \to \mathbb{R}$ continuous at d such that

$$g(y) - g(d) = \psi(y)(y-d) \quad \forall\, y \in I$$

and $\psi(d) = g'(d)$.

For $x \in J$, substituting $y = f(x)$ & $d = f(c)$, we have

$$g(f(x)) - g(f(c)) = \psi(f(x))(f(x) - f(c))$$

$$\therefore \quad g\circ f(x) - g\circ f(c) = \psi(f(x))\varphi(x)(x-c)$$

$$= \left[(\psi\circ f)(x)\,\varphi(x)\right](x-c), \quad \forall\, x \in J$$

Since f diff. at c, f is continuous at c.

Together with ψ is continuous at $f(c) = d$, we have

$\psi\circ f$ is continuous at c.

Therefore $(\psi\circ f)(x)\,\varphi(x)$ is continuous at c (as φ is continuous at c)

$\therefore$ $g\circ f$ is differentiable at c by Carathéodory's Thm

and $(g\circ f)'(c) = (\psi\circ f)(c)\,\varphi(c) = \psi(d) f'(c) = g'(d) f'(c)$

$$= g'(f(c))f'(c). \quad \#$$

<u>Note</u>: By using Carathéodory's Thm 6.1.5, we avoided the discussion of whether $f(x) - f(c) = 0$ as in the usual proof by the algebraic expression

$$\frac{g(f(x)) - g(f(c))}{x-c} = \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x-c}.$$

eg 6.1.7 let $f: I \to \mathbb{R}$ is differentiable on I (ie at all points of I)

(a) Chain rule (also) $\Rightarrow$ $(f^n)'(x) = n\,(f(x))^{n-1} f'(x)$

(b) If further assume $f(x) \neq 0$, $\forall x \in I$, (mistake in textbook, $f' \neq 0$ not needed)

$$\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{(f(x))^2}, \quad \forall x \in I$$

by using $g(y) = \frac{1}{y}$ for $y \neq 0$ and $g'(y) = -\frac{1}{y^2}$, $\forall y \neq 0$.

(c)
$$|f|'(x) = \operatorname{sgn}(f(x)) \cdot f'(x) = \begin{cases} f'(x), & \text{if } f(x) > 0 \\ -f'(x), & \text{if } f(x) < 0 \end{cases}$$

(where $\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ the signum function)

Pf: Consider $g(x) = |x|$. Then $g: (-\infty, \infty) \to \mathbb{R}$

and we've proved that g is differentiable at $x \neq 0$.

$$g'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$\therefore$ For $x \neq 0$, $g'(x) = \operatorname{sgn}(x)$

(but $g' \neq \operatorname{sgn}$ at $x = 0$, because $g'(0)$ doesn't exist)

Therefore, by chain rule,

$|f|(x)$ is differentiable at x where $f(x) \neq 0$

and $\qquad |f|'(x) = g'(f(x))\, f'(x) = \operatorname{sgn}(f(x))\, f'(x)$

$$= \begin{cases} f'(x), & f(x) > 0 \\ -f'(x), & f'(x) < 0 . \end{cases}$$

(At x where $f(x)=0$, the situation is more complicated:

(i) if $f(x)=x^2$, then $|f|(x) = x^2$ is differentiable also at $x=0$

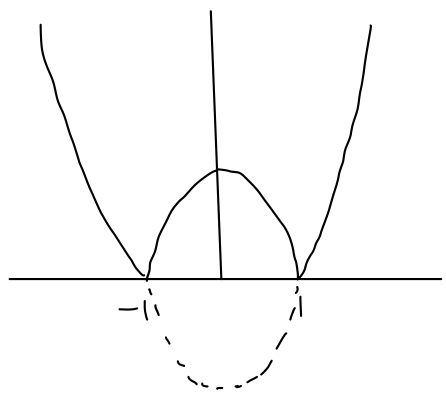
(ii) if $f(x)=x$, then $|f|(x) = |x|$ is not differentiable at $x=0$

See exercise 7 of §6.1 on page 171 of the text book.)

Concrete example: $f(x) = x^2-1$, then $f(x)=0 \Leftrightarrow x=\pm 1$.

$\therefore \quad |f|(x) = |x^2-1|$ is differentiable for $x \neq \pm 1$ and

$$\frac{d}{dx}|x^2-1| = |f|'(x) = \operatorname{sgn}(x^2-1)\cdot 2x = \begin{cases} 2x, & \text{if } x<-1 \text{ or } x>1 \\ -2x, & \text{if } -1<x<1 \end{cases}$$



(d) <u>Derivatives of trigonometric functions</u>.

Let $S(x) = \sin x$, $C(x) = \cos x$ for $x \in \mathbb{R}$.

We'll define these two functions and prove the following later in section 8.4:

$$S'(x) = \cos x = C(x), \quad C'(x) = -\sin x = -S(x).$$

Using these facts & quotient rule, we have the formula for derivatives of other trigonometric functions:

$$\left.\begin{aligned} D \tan x &= (\sec x)^2 \\ D \sec x &= (\sec x)(\tan x) \end{aligned}\right\} \quad \text{for } x \neq \frac{(2k+1)\pi}{2}, \ k \in \mathbb{Z}$$

$$\left.\begin{aligned} D \cot x &= -(\csc x)^2 \\ D \csc x &= -(\csc x)(\cot x) \end{aligned}\right\} \quad \text{for } x \neq k\pi, \ k \in \mathbb{Z}$$

(e) $$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

By Chain rule, (product rule & quotient rule,) for $x \neq 0$

$$f'(x) = 2x \sin\left(\tfrac{1}{x}\right) - \cos\left(\tfrac{1}{x}\right) \qquad (\text{check!})$$

But at $x=0$, we must use definition of derivative to

find $$f'(0) = \lim_{\substack{x \to 0 \\ (x \neq 0)}} \frac{f(x) - f(0)}{x - 0} = \lim_{\substack{x \to 0 \\ (x \neq 0)}} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{\substack{x \to 0 \\ (x \neq 0)}} x \sin\frac{1}{x} = 0$$

$\therefore$ $f'(x)$ exists for all $x \in \mathbb{R}$ and

$$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(Note: However, $f'(x)$ is <u>discontinuous</u> at $x = 0$

as $\displaystyle\lim_{\substack{x \to 0 \\ (x \neq 0)}} \left(2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right)$ doesn't exist. (check)

$\therefore$ f differentiable $\forall x \not\Rightarrow f'$ is continuous.)