MATH 2060A Mathematical Analysis II
2024-25 Term 1
Suggested Solution to Homework 9

9.4-1 Discuss the convergence and the uniform convergence of the series $\sum f_n$, where $f_n(x)$ is given by:

(a) $(x^2 + n^2)^{-1}$, (c) $\sin(x/n^2)$.

Solution. (a) Note that $|f_n(x)| = |(x^2 + n^2)^{-1}| \le \frac{1}{n^2}$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$. Moreover, $\sum \frac{1}{n^2}$ is convergent. By Weierstrass M-Test, $\sum f_n$ is uniformly convergent on $\mathbb{R}$.

(c) Let $a > 0$. Then $|f_n(x)| = |\sin(x/n^2)| \le |x/n^2| \le \frac{a}{n^2}$ for $x \in [-a, a]$, $n \in \mathbb{N}$. Moreover, $\sum \frac{a}{n^2}$ is convergent. By Weierstrass M-Test, $\sum f_n$ is uniformly convergent on $[-a, a]$. Since $a > 0$ is arbitrary, $\sum f_n$ is convergent on $\mathbb{R}$.

However, $\sum f_n$ is not uniformly convergent on $\mathbb{R}$. Take $\varepsilon_0 = 1$. Then for any $n \in \mathbb{N}$,

$$|f_n(n^2\pi/2)| = |\sin \pi/2| = 1 = \varepsilon_0.$$

By Cauchy Criterion 9.4.5, $\sum f_n$ is not uniformly convergent on $\mathbb{R}$.

□

9.4-2 If $\sum a_n$ is an absolutely convergent series, then the series $\sum a_n \sin nx$ is absolutely and uniformly convergent.

Solution. Let $\varepsilon > 0$. Since $\sum a_n$ is absolutely convergent, there is $M \in \mathbb{N}$ such that if $m > n \ge M$, then

$$|a_{n+1}| + |a_{n+2}| + \cdots + |a_m| < \varepsilon.$$

Now, if $m > n \ge M$, then

$$|a_n \sin nx| + |a_{n+1} \sin(n+1)x| + \cdots + |a_m \sin mx| \le |a_{n+1}| + |a_{n+2}| + \cdots + |a_m| < \varepsilon.$$

By Cauchy Criterion 9.4.5, $\sum a_n \sin nx$ is absolutely and uniformly convergent on $\mathbb{R}$. □