MATH 2060A Mathematical Analysis II
2024-25 Term 1
Suggested Solution to Homework 8

9.1-7 (a) If $\sum a_n$ is absolutely convergent and (b_n) is a bounded sequence, show that $\sum a_n b_n$ is absolutely convergent.

(b) Give an example to show that if the convergence of $\sum a_n$ is conditional and (b_n) is a bounded sequence, then $\sum a_n b_n$ may diverge.

Solution. (a) Let $B > 0$ be a bound of (b_n). Then, for any $N \geq 1$,

$$\sum_{n=1}^{N} |a_n b_n| \leq \sum_{n=1}^{N} B|a_n| \leq B \sum_{n=1}^{N} |a_n|.$$

Hence the absolute convergence of $\sum a_n$ implies the absolute convergence of $\sum a_n b_n$.

(b) Consider $a_n = (-1)^n/n$ and $b_n = (-1)^n$. Then $\sum a_n$ is conditionally convergent and (b_n) is bounded by 1. However $\sum a_n b_n = \sum \frac{1}{n}$ is divergent.

□

9.1-13 (a) Does the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}} \right)$ converge?

(b) Does the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+1} - \sqrt{n}}{n} \right)$ converge?

Solution. (a) Note that, for $n \in \mathbb{N}$,

$$\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}} = \frac{1}{\sqrt{n}(\sqrt{n+1} + \sqrt{n})} \geq \frac{1}{2(n+1)} \geq 0.$$

Since $\sum \frac{1}{2(n+1)}$ is divergent, it follows from the Comparison Test 3.7.7 that $\sum \left(\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}} \right)$ is also divergent.

(b) Note that, for $n \in \mathbb{N}$,

$$0 \leq \frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} \leq \frac{1}{n^{3/2}}.$$

Since $\sum \frac{1}{n^{3/2}}$ is convergent, it follows from the Comparison Test 3.7.7 that $\sum \left(\frac{\sqrt{n+1} - \sqrt{n}}{n} \right)$ is also convergent.

□

9.2-3 Discuss the convergence or the divergence of the series with nth term (for sufficiently large n) given by:

(a) $(\ln n)^{-p}$, (b) $(\ln n)^{-n}$.

Solution. (a) Clearly, the series is divergent if $p \le 0$ by the n-th Term Test.

Suppose $p > 0$. Note that

$$r := \lim \left| \frac{1/n}{(\ln n)^{-p}} \right| = \lim \frac{(\ln n)^p}{n} = 0.$$

Since $\sum \frac{1}{n}$ is divergent, it follows from Limit Comparison Test 3.7.8 that $\sum (\ln n)^{-p}$ is also divergent.

(b) Note that

$$r := \lim |(\ln n)^{-n}|^{1/n} = \lim (\ln n)^{-1} = 0 < 1.$$

By (Corollary 9.2.3 of) the Root Test, $\sum (\ln n)^{-n}$ is absolutely convergent, hence convergent.

$\square$