MATH 2060A Mathematical Analysis II
2024-25 Term 1
Suggested Solution to Homework 7

8.1-14 Show that if $0 < b < 1$, then the convergence of the sequence in Exercise 4 is uniform on the interval $[0, b]$, but is not uniform on the interval $[0, 1]$.

Solution. Let (f_n) be the sequence of functions considered in Exercise 4, and let f be its limit. Since $0 \le f_n(x) = \frac{x^n}{1+x^n} \le \frac{b^n}{1+0} = b^n$ for any $x \in [0, b]$, we have

$$\|f_n - 0\|_{[0,b]} \le b^n \quad \text{for all } n \in \mathbb{N}.$$

As $0 < b < 1$, we have $\lim(b^n) = 0$ and so $\lim \|f_n - 0\|_{[0,b]} = 0$. Therefore (f_n) converges uniformly to $f \equiv 0$ on $[0, b]$.

On the other hand, for all $n \in \mathbb{N}$,

$$\|f_n - f\|_{[0,1]} \ge |f_n(2^{-1/n}) - f(2^{-1/n})| = \left| \frac{1/2}{1 + 1/2} - 0 \right| = \frac{1}{3}.$$

So $\|f_n - f\|_{[0,1]} \not\to 0$ as $n \to \infty$. Therefore (f_n) does not converge uniformly to f on $[0, 1]$. □

8.1-22 Show that if $f_n(x) := x + 1/n$ and $f(x) := x$ for $x \in \mathbb{R}$, then (f_n) converges uniformly on $\mathbb{R}$ to f, but the sequence (f_n^2) does not converge uniformly on $\mathbb{R}$. (Thus the product of uniformly convergent sequences of functions may not converge uniformly.)

Solution. Since $\|f_n - f\|_{\mathbb{R}} = 1/n \to 0$ as $n \to \infty$, (f_n) converges uniformly on $\mathbb{R}$ to f.

On the other hand, for all $n \in \mathbb{N}$,

$$f_n^2(x) - f^2(x) = \left(x + \frac{1}{n}\right)^2 - x^2 = \frac{2x}{n} + \frac{1}{n^2} \quad \text{for any } x \in \mathbb{R},$$

so that

$$\|f_n^2 - f^2\|_{\mathbb{R}} \ge |f_n^2(n) - f^2(n)| = 2 + \frac{1}{n^2} \ge 2.$$

Therefore (f_n^2) does not converge uniformly to f^2 on $\mathbb{R}$. And so (f_n^2) does not converge uniformly on $\mathbb{R}$ because f^2 is the pointwise limit of (f_n^2). □

8.1-23 Let $f(_n)$, (g_n) be sequences of bounded functions on A that converge uniformly on A to f, g, respectively. Show that $(f_n g_n)$ converges uniformly on A to fg.

Solution. Since $f(_n)$, (g_n) converge uniformly on A to f, g, respectively, we have for any $\varepsilon > 0$, there is $N = N_\varepsilon \in \mathbb{N}$ such that if $n \ge N$, then

$$\|f_n - f\|_A < \varepsilon \quad \text{and} \quad \|g_n - g\|_A < \varepsilon. \tag{1}$$

First we show that f, g are bounded on A. By taking $\varepsilon = 1$, there is $N_1 \in \mathbb{N}$ such that if $x \in A$, then

$$|f(x)| \le |f_{N_1}(x)| + |f_{N_1}(x) - f(x)| \le \|f_{N_1}\|_A + 1,$$

and

$$|f_n(x)| \leq |f(x)| + |f_n(x) - f(x)| \leq \|f_{N_1}\|_A + 2, \qquad \text{for } n \geq N_1.$$

Thus we can find $M > 0$ such that $\|f\|_A, \|f_n\|_A \leq M$ for all $n \geq N_1$. Similarly we can find $M' > 0$ such that $\|g\|_A, \|g_n\|_A \leq M'$ for all $n \geq N_1$.

Now, by applying (1) to an arbitrary $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that if $n \geq N_\varepsilon$ and $x \in A$, we have

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\ &\leq M\varepsilon + M'\varepsilon. \end{aligned}$$

So $\|f_n g_n - fg\|_A \leq (M + M')\varepsilon$ for all $n \geq N_\varepsilon$. Therefore $(f_n g_n)$ converges uniformly on A to fg. $\square$