Recall: Lipschitz fcn:

$f: A \to \mathbb{R}$ is lipschitz cts if $\exists \Lambda > 0$ s.t.

$|f(x) - f(y)| \leq \Lambda \cdot |x - y|$, $\forall x, y \in A$.

From discussion Last time:

Lipschitz fcn $\Rightarrow$ unif. cts.

($\forall \varepsilon > 0$, choose $\delta = \varepsilon / \Lambda > 0$
s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$)

$\Leftarrow$: false in general.

eg. 1. $f: [0,1] \to \mathbb{R}$ given by $f(x) = \sqrt{x}$

s.t. f = unif. cts using ε-δ argument
or unif. continuity thm.

$$\left| \frac{f(x) - f(0)}{x - 0} \right| = \frac{1}{\sqrt{x}} \to \infty \text{ as } x \to 0^+.$$

2. $f: [0,1] \to \mathbb{R}$ given by $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases}$

is also unif cts since. $\sin(\frac{1}{x})$ is bdd, ...

But f is Not lipschitz:

Choose $x_n = \frac{1}{2n\pi + \pi/2}$, $y_n = \frac{1}{2n\pi}$ s.t.

$f(x_n) = x_n \cdot 1$, $f(y_n) = 0$.

$$\frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = \frac{|x_n|}{|x_n - y_n|}$$

$$|x_n - y_n| = \frac{1}{2n\pi} \cdot \frac{1}{2n\pi + \frac{\pi}{2}} \left(\frac{\pi}{2}\right).$$

$$\Rightarrow \frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = \frac{|x_n|}{|x_n - y_n|} = \frac{(2n\pi + \frac{\pi}{2})^{-1}}{\frac{1}{2n\pi} \cdot \frac{1}{2n\pi + \frac{\pi}{2}} \cdot \frac{\pi}{2}}$$

$$= 4n \rightarrow +\infty \quad \text{as} \quad n \to +\infty.$$

$\Rightarrow f \neq$ Lipschitz. (i.e. $\nexists\, \Lambda > 0$ s.t. $\frac{|f(x) - f(y)|}{|x-y|} \leq \Lambda$)

Let $X \subsetneq \mathbb{R}$, $f : X \to \mathbb{R}$ lipschitz fcn. on X.

Y

i.e. $X \subseteq Y \subseteq \mathbb{R}$.

Q: Ask for a reasonable extension of f over Y.
(Best one??)

Looking for $F : Y \to \mathbb{R}$ s.t.

① $F|_X = f$ ②': F is "as Lip as" f.

② F = Lip.

Defn Given $f : A \to \mathbb{R}$, which is Lipschitz.

then Lipschitz constant of f is defined to be

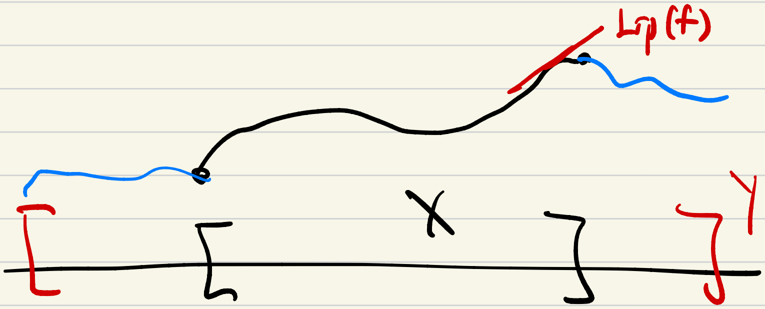
$$\mathrm{Lip}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} \;\middle|\; x, y \in A,\ x \neq y \right\}.$$

$$\boxed{②': \ \mathrm{Lip}(F) = \mathrm{Lip}(f).}$$

prop: Let $X \subseteq Y \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ is a Lipschitz fcn. Then $\exists\, F : Y \to \mathbb{R}$, Lipschitz s.t.

① $F|_X = f$ ② $\mathrm{Lip}(F) = \mathrm{Lip}(f)$.

pf:



Fix $x \in X$, $y \in Y$

$$F(y) := \inf\{ f(x) + \mathrm{Lip}(f)\cdot|x-y| : x \in X \}$$

claim: $F(x_0) = f(x_0)$ if $x_0 \in X$.

pf: $\forall x \in X$

$$f(x_0) \le f(x) + \mathrm{Lip}(f)\cdot|x - \boxed{x_0}|. \quad \text{(by def)}$$

$$\overset{\inf}{\Rightarrow} \quad f(x_0) \le F(x_0).$$

$$F(x_0) \overset{\text{(by defn)}}{\le} f(x) + \mathrm{Lip}(f)\cdot|x - x_0|, \quad \forall x \in X$$

Putting $x = x_0 \in X \Rightarrow F(x_0) \le f(x_0)$

$$\Rightarrow \quad F(x_0) = f(x_0) \quad \#$$

Claim: $\mathrm{Lip}(F) = \mathrm{Lip}(f)$.

pf: Let $y_1, y_2 \in Y$.

$$F(y_1) \leq f(x) + \mathrm{Lip}(f) \cdot |x - y_1|$$

$$\leq \underline{f(x) + \mathrm{Lip}(f) \cdot |x - y_2|} + \mathrm{Lip}(f) \cdot |y_1 - y_2|$$

taking inf

$$\Rightarrow F(y_1) \leq F(y_2) + \mathrm{Lip}(f) \cdot |y_1 - y_2|$$

$\because$ y_1, y_2 are arbitrary,

$$\Rightarrow |F(y_1) - F(y_2)| \leq \mathrm{Lip}(f) \cdot |y_1 - y_2|$$

$$\Rightarrow \mathrm{Lip}(F) \leq \mathrm{Lip}(f).$$

$$\mathrm{Lip}(F) = \sup \left\{ \frac{|F(y_1) - F(y_2)|}{|y_1 - y_2|} : y_1 \neq y_2 \in Y \right\}$$

$\because$ $X \subseteq Y$, $F|_X = f$.

$\therefore$ $\mathrm{Lip}(F) \geq \mathrm{Lip}(f)$

$$\Rightarrow \boxed{\mathrm{Lip}(F) = \mathrm{Lip}(f)} \quad \#$$

Defn: A Lipschitz fcn $f: A \to \mathbb{R}$ is said to be contraction if $Lip(f) < 1$.

i.e., $|f(x) - f(y)| \le \sigma \cdot |x-y|$ for some $\sigma \in (0,1)$

Prop: If A = closed, non-empty subset of $\mathbb{R}$, $f: A \to A$ is a contraction, then $\exists!\, a \in A$ s.t. $f(a) = a$.

pf: Uniqueness: if $a, b \in A$ s.t.

$$\begin{cases} f(a) = a \\ f(b) = b \end{cases}$$

contraction

then $|f(a) - f(b)| = |a-b| \le \sigma \cdot |a-b|$ for some $\sigma \in (0,1)$

$\Rightarrow (1-\sigma)|a-b| \le 0 \Rightarrow a = b$.

Existence: Fix $x_1 \in A$, $x_2 = f(x_1)$

Construct (x_n) by $f(x_n) = x_{n+1}$, $\forall n \in \mathbb{N}$.

$$|x_{n+2} - x_{n+1}| = |f(x_{n+1}) - f(x_n)|$$
$$\le \sigma \cdot |x_{n+1} - x_n|, \quad \forall n \in \mathbb{N}.$$

$$\Rightarrow |x_{n+1} - x_n| \le \sigma^{n-1} \cdot |x_2 - x_1|, \quad \forall n \in \mathbb{N}.$$

Δ-ineq $\Rightarrow |x_{n+p} - x_n| \le \underbrace{\boxed{|x_2 - x_1|}}_{\text{const}} \cdot \boxed{\sum_{i=n}^{n+p-1} \sigma^{i-1}}$

check easily by Geometric series formula :

$\forall \varepsilon > 0$, $\exists N$ s.t. if $n > N$, $p \in \mathbb{N}$,

$$\sum_{i=n}^{n+p-1} r^{i-1} < \varepsilon .$$

$$\Rightarrow \quad (x_n) \text{ is cauchy} \overset{\text{cauchy crit.}}{\Longrightarrow} (x_n) \text{ is convergent}$$

$$\Rightarrow \quad x_n \to \bar{x} \text{ for some } \bar{x} \in \bar{A} \overset{\text{closed.}}{=} A.$$

$\because$ f is cts at $\bar{x}$ and $f(x_n) = x_{n+1}$

$$\therefore \quad \lim_{n\to\infty} f(x_n) \underset{\text{seq crit.}}{=} f(\lim x_n) = f(\bar{x})$$

$$\| $$

$$\lim_{n\to\infty} x_{n+1} = \bar{x} .$$

i.e. $\bar{x}$ = fixed pt of $f: A \to \mathbb{R}$.

<u>Rmk</u>: $\mathrm{Lip}(f) < 1$ is necessary

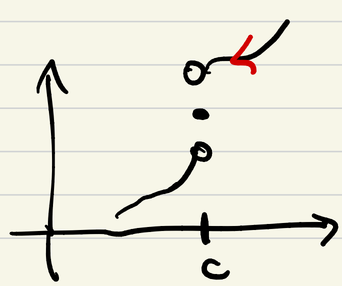
eg $f(x) = x-1$ has no fixed pt. on $\mathbb{R}$ closed. $(\bar{\mathbb{R}} = \mathbb{R})$

Monotone function :

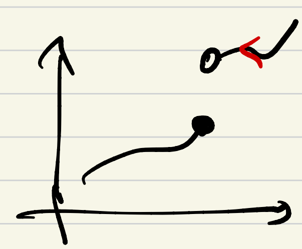
WLOG, consider increasing function :

$f: A \to \mathbb{R}$ s.t. $a \geq b \Rightarrow f(a) \geq f(b)$.

eg.



or



At $c \in D^{+}(A) \cap A$,

(i.e. $\exists\, a_i \in A \setminus \{c\}$ s.t. $a_i > c$ and $a_i \to c$.)

Prop: $\lim_{x \to c^+} f(x)$ exists and equals to

$$L(c) := \inf \{ f(x) \mid x \in A,\ x > c \}$$

pf: $\because c \in D^{+}(A)$ $\therefore \{ f(x) \mid x \in A,\ x > c \} \neq \phi$.

$\because c \in A$ $\therefore f(x) \geq f(c)$, $\forall x > c$.

$\Rightarrow L(c)$ exists by completeness.

Claim: $f(x) \to L(c)$ as $x \to c^+$.

pf: Fix $\varepsilon > 0$, $\because L(c) = \inf \{ \cdots \}$

$\therefore \exists\, x_1 \in A,\ x_1 > c$ s.t. $L(c) + \varepsilon > f(x_1)$

taking $\delta > 0$ small enought s.t. $0 < \delta < x_1 - c$

then $\forall\, x \in A \cap (c, c+\delta)$,

$c < x < c + \delta < x_1$

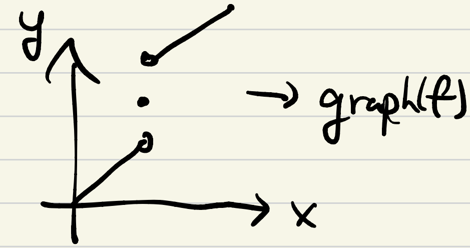
$\Rightarrow \underset{\substack{\text{VI} \\ L(c)}}{f(x)} \leq f(x_1) < L(c) + \varepsilon$.

(defn)

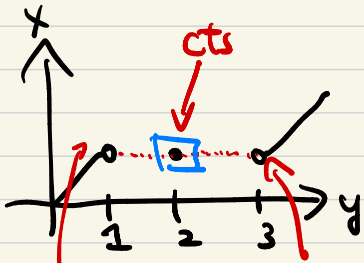
#.

Eg:

$$f(x) = \begin{cases} x & \text{if } 0<x<1 \\ 2 & \text{if } x=1 \\ x+2 & \text{if } 1<x<2 \end{cases}$$



$$g(y) = f^{-1}(y) = \begin{cases} y & \text{if } 0<y<1 \\ 1 & \text{if } y=2 \\ y-2 & \text{if } 3<y<4 \end{cases}$$



graph(g)

$$\lim_{y \to 1^-} g(y) = 2 = \lim_{y \to 3^+} g(y)$$

$$2 = g(d) = \lim_{y \to d^+} g(y) = \lim_{y \to d^-} g(y)$$

prop: $f: A \to \mathbb{R}$ is strictly increasing $(a>b \Rightarrow f(a)>f(b))$ and A = interval, then $g = f^{-1}: f(A) \to A$ exists. And if $d \in f(A)$, then $g(d) = L(d)$ provided that $L(d)$ exists.

$\big(= l(d)$, left hand limit resp.$\big)$

pf: The existence of g follows from injective of f. (by strictly $\uparrow$)

Remains to show $L(d) = g(d)$ (if $L(d)$ exists)

By defn, $L(d) \geq g(d)$ since $g \uparrow$ (by previous prop)

Suppose $L(d) > g(d)$

let $\varepsilon > 0$, $\exists\, y_1 \in f(A)$ (Domain of g) s.t.

$y_1 > d$ and $L(d) + \varepsilon > g(y_1) \geq L(d) > g(d)$.

Consider $g(d), g(y_1) \in A$

$\Rightarrow (g(d), g(y_1)) \subseteq A$ since A = interval.

$\forall z \in (g(d), g(y_1))$,

$f(g(y_1)) = y_1 > y = f(z) > \underline{d} = f(g(d))$

$\Rightarrow z = g(f(z)) = g(y) \geq L(d)$

let $z \to g(d) \Rightarrow g(d) \geq L(d) > g(d)$. $\rightarrow\leftarrow$.

$\therefore g(d) = L(d)$ #

Hence, $g: f(A) \to A$ is cts

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prop: $f: [a,b] \to \mathbb{R}$, monotone (WLOG assume $\uparrow$)

Then the set $\{c \mid f$ is Not cts at $c\} = D$

is countable set.

pf: $\because f \uparrow$

At each $x_0 \in [a,b]$,

might define $j(x) = \lim_{x \to x_0^+} f(x) - \lim_{x \to x_0^-} f(x)$

if $x_0 = a$, $\lim_{x \to x_0^-} f(x) := f(a)$.

if $x_0 = b$, $\lim_{x \to x_0^+} f(x) := f(b)$.

By $\uparrow$, $j(x) \geq 0 \quad \forall x \in [a,b]$.

WLOG, assume $\underline{f(b) > f(a)}$

otherwise $f(x) \equiv f(a)$ on $[a,b]$.

$\therefore \forall \quad a < x_1 < x_2 < \cdots < x_N \leq b$, we have

$f(a) \leq f(a) + j(x_1) + j(x_2) + \cdots + j(x_N) \leq f(b)$. (Exercise)

$$D_n = \left\{ c \;\middle|\; j(c) > \frac{f(b)-f(a)}{n} \right\}$$

$$\Rightarrow \quad D = \bigcup_{n \geq 1} \boxed{D_n} \quad (x \in D \text{ iff } j(x) > 0)$$

(Ex)

focus on this.

claim: $|D_n| \leq n$.

since otherwise $\exists \; x_1 < \cdots < x_{n+1}$ in D_n s.t.

$$f(b) \geq f(a) + j(x_1) + \cdots + j(x_{n+1}) \geq f(a) + \frac{n+1}{n}(f(b)-f(a))$$

$\rightarrow\leftarrow$.

$\therefore \; D = \bigcup_{n=1}^{\infty} D_n$ and $|D_n| \le n$

$\Rightarrow D =$ countable. #