**Math 2058, HW 4. Due: 15 Nov 2024, before 11:59 pm**

(1) Show that $f : A \to \mathbb{R}$ is continuous if and only if $f^{-1}(C)$ is closed in $A$ for all $C$ which is closed in $f(A)$.

(2) Let $f : (0, +\infty) \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} \dfrac{1}{m^2}, & \text{if } x = \dfrac{m}{n},\ \gcd(m,n) = 1; \\ \\ 0 & \textit{otherwise.} \end{cases}$$

Determine the set of $c$ where $f$ is continuous at $c$.

(3) Let $A$ be a subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ be a continuous function. Show that the set of $c$ where $f(c) = 0$, is closed in $A$.

(4) Let $A$ be a subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ be a continuous function. Show that

$$f(\overline{E}) \subset \overline{f(E)}$$

for all $E \subset A$. Show that it can be a proper subset by giving an example.

(5) Let $E$ be a closed subset in $\mathbb{R}$ and $f : E \to \mathbb{R}$ be a continuous function. Show that there exists $g : \mathbb{R} \to \mathbb{R}$ such that $g|_E = f$ and $g$ is continuous. Is it true if $E$ is not closed? Justify your answer.

(6) Show that $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{x}{x^2 + 2}$$

is continuous.