THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I
Suggested Solutions for HW5

1. Show that $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{1}{x^2+1}$$

is uniformly continuous by using $\varepsilon - \delta$ terminology.

Solution. Let $\varepsilon > 0$ be given. For $x, y \in \mathbb{R}$, we have that $x^2+1, y^2+1 \geq 1$ and

$$\begin{aligned}
\left|\frac{1}{x^2+1} - \frac{1}{y^2+1}\right| = \left|\frac{y^2-x^2}{(x^2+1)(y^2+1)}\right| &= |y-x|\left|\frac{y+x}{(x^2+1)(y^2+1)}\right| \\
&\leq |y-x|\left(\left|\frac{x}{(x^2+1)(y^2+1)}\right| + \left|\frac{y}{(x^2+1)(y^2+1)}\right|\right) \\
&\leq |y-x|\left(\left|\frac{x}{x^2+1}\right| + \left|\frac{y}{y^2+1}\right|\right) \\
&\leq 2|y-x|
\end{aligned}$$

where we have used the fact that $\dfrac{x}{x^2+1} \leq 1$ for all $x \in \mathbb{R}$. Then, taking $\delta := \dfrac{\varepsilon}{2}$, we obtain the desired result. ◀

2. Suppose $f : [0, +\infty) \to \mathbb{R}$ is a continuous function such that $f|_{[a,+\infty)}$ is uniformly continuous for some $a > 0$. Show that f is uniformly continuous.

Solution. Let $\varepsilon > 0$ be given. By uniform continuity of f on $[a, +\infty)$, we know that there is a $\delta_1 > 0$ such that if $x, y \in [a, +\infty)$ with $|x - y| < \delta_1$, we have that $|f(x) - f(y)| < \varepsilon$. Since f is continuous, it is uniformly continuous on $[0, 2a]$, that is, there is a $\delta_2 > 0$ such that if $x, y \in [0, 2a]$ with $|x - y| < \delta_2$, then $|f(x) - f(y)| < \varepsilon$. Set $\delta := \min\{\delta_1, \delta_2, a\}$. Let $x, y \in \mathbb{R}$ with $|x - y| < \delta$. We have a few cases to consider.

Suppose first that at least one of $x, y \in [0, a]$, say x. Then the triangle inequality guarantees that $y \in [0, 2a]$ and we have that $|f(x) - f(y)| < \varepsilon$ as required.

Now suppose that $x \in [a, +\infty)$ while $y \in [0, a)$. Then triangle inequality again guarantees that $x \in [0, 2a]$, and we have that $|f(x) - f(y)| < \varepsilon$ as required.

The other cases are trivial, and we are done. ◀

3. If $f, g : \mathbb{R} \to \mathbb{R}$ are two uniform continuous function, show that $f \circ g$ is also uniform continuous.

Solution. Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there is a $\delta_1 > 0$ such that if $x, y \in \mathbb{R}$ with $|x - y| < \delta_1$, we have that $|f(x) - f(y)| < \varepsilon$. Since g is uniformly continuous, there is a $\delta_2 > 0$ such that if $x, y \in \mathbb{R}$ with $|x-y| < \delta_2$, we have that $|g(x) - g(y)| < \delta_1$. Then with $|x - y| < \delta_2$, we have that $|f(g(x)) - f(g(y))| < \varepsilon$ as required. ◀

4. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\lim_{x\to+\infty} f(x) = L_1, \lim_{x\to-\infty} f(x) = L_2$$

for some L_i. Show that there exists $\overline{x} \in \mathbb{R}$ such that $f(\overline{x}) \geq f(x)$ for all $x \in \mathbb{R}$ if $f(0) > \max\{L_1, L_2\}$.

Solution. Set $\varepsilon := f(0) - \max\{L_1, L_2\} > 0$. By the assumption about limits at infinity, we have that there are $M_1, M_2 > 0$ such that for all $x \geq M_1$ and $x \leq -M_2$,

$$\begin{aligned} |f(x) - L_1| < \varepsilon \Rightarrow f(x) - L_1 < f(0) - \max\{L_1, L_2\} \Rightarrow f(x) < f(0), \\ |f(x) - L_2| < \varepsilon \Rightarrow f(x) - L_2 < f(0) - \max\{L_1, L_2\} \Rightarrow f(x) < f(0). \end{aligned}$$

So $f(0) \geq f(x)$ for all $x \in (-\infty, -M_2] \cup [M_1, +\infty)$. Since f is continuous, restricted to $[-M_2, M_1]$, it attains maximum at some point $x_0 \in [-M_2, M_1]$. Then if $f(0) > f(x_0)$, taking $\overline{x} = 0$ yields the desired point. If $f(x_0) > f(0)$, then taking $\overline{x} = x_0$ yields the desired point. ◀

5. Let A be a compact set in $\mathbb{R}$. Suppose $f : A \to \mathbb{R}$ is a real valued function such that for any $\varepsilon > 0$, there is a polynomial g_ε such that $\sup_A |f(x) - g_\varepsilon(x)| < \varepsilon$. Show that f is uniformly continuous.

Solution. Let $\varepsilon > 0$ be given and let $x, y \in A$. Then there is a polynomial $g_{\varepsilon/3}$ such that for all $x \in A$, $|f(x) - g_{\varepsilon/3}(x)| < \dfrac{\varepsilon}{3}$. Since $g_{\varepsilon/3}$ is a polynomial, it is continuous and hence uniformly continuous on A, that is, there is a $\delta > 0$ such that for $|y - x| < \delta$, $|g_{\varepsilon/3}(x) - g_{\varepsilon/3}(y)| < \dfrac{\varepsilon}{3}$. Then by the triangle inequality, we have that for $|y - x| < \delta$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - g_{\varepsilon/3}(x)| + |g_{\varepsilon/3}(x) - g_{\varepsilon/3}(y)| + |g_{\varepsilon/3}(x) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

as required. ◀

6. Suppose $f : (0, 1] \to \mathbb{R}$ is a bounded continuous function. Show that the function given by $g(x) = xf(x)$ is uniformly continuous on $(0, 1)$.

Solution. Let $x, y \in (0, 1)$. Since f is bounded on $(0, 1]$, there is an $M \geq 0$ such that for all $x \in (0, 1)$, $|f(x)| \leq M$. We see that

$$\begin{aligned} |g(x) - g(y)| &= |xf(x) - yf(y)| = |xf(x) - yf(x) + yf(x) + yf(x) - yf(y)| \\ &\leq |f(x)||x - y| + |y||f(x) - f(y)| \leq M|x - y| + |y||f(x) - f(y)|. \end{aligned}$$

Since f is continuous on $(0, 1]$, it is uniformly continuous on $\left[\frac{\varepsilon}{4M}, 1\right]$, that is, there is a $\delta_1 > 0$ such that if $x, y \in \left[\frac{\varepsilon}{4M}, 1\right]$ with $|x - y| < \delta_1$, then $|f(x) - f(y)| < \dfrac{\varepsilon}{2}$. Then taking $\delta = \min\{\delta_1, \frac{\varepsilon}{2M}\}$, we see by above that if $x, y \in \left[\frac{\varepsilon}{4M}, 1\right)$ with $|x - y| < \delta$, then

$$|g(x) - g(y)| \leq M|x - y| + |y||f(x) - f(y)| < M \cdot \frac{\varepsilon}{2M} + 1 \cdot \frac{\varepsilon}{2} + \varepsilon.$$

If $y \in \left(0, \frac{\varepsilon}{4M}\right)$, then by above we have that

$$\begin{aligned} |g(x) - g(y)| &\leq M|x - y| + |y||f(x) - f(y)| < M|x - y| + |y|(|f(x)| + |f(y)|) \\ &< M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{4M} \cdot 2M < \varepsilon. \end{aligned}$$

If $x \in \left(0, \frac{\varepsilon}{4M}\right)$, then interchanging the roles of x and y in the same estimate above and repeating the argument yields the desired result. ◀