THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I
Suggested Solutions for HW4

1. Show that $f : A \to \mathbb{R}$ is continuous if and only if $f^{-1}(C)$ is closed in A for all C which is closed in $f(A)$.

Solution. Suppose f is continuous and C is closed in $f(A)$. Let (x_n) be a convergent sequence in $f^{-1}(C)$ with $x := \lim_{n\to\infty} x_n \in A$. It suffices to show that $x \in f^{-1}(C)$. Since f is continuous at x, we have that the sequence $(f(x_n)) \subset C$ converges to $f(x)$, and since C is closed in A, we know that $f(x) \in C$. Hence, $x \in f^{-1}(C)$ as required, and we have that $f^{-1}(C)$ is closed in A.

Now suppose that $f^{-1}(C)$ is closed in A for all C which is closed in $f(A)$. Let U be open in $f(A)$. Then $f(A) \setminus U$ is closed in $f(A)$ and by assumption $f^{-1}(f(A) \setminus U)$ is closed in A. But we have $f^{-1}(f(A) \setminus U) = A \setminus f^{-1}(U)$, which means $f^{-1}(U)$ is open in A, and hence f is continuous. ◀

2. Let $f : (0, +\infty) \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} \frac{1}{m^2}, & \text{if } x = \frac{m}{n}, \gcd(m, n) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Determine the set of c where f is continuous at c.

Solution. We will show that f is continuous on $(0, +\infty) \setminus \mathbb{Q}$. We first show that f is not continuous on $(0, +\infty) \cap \mathbb{Q}$. Let $x \in (0, +\infty) \cap \mathbb{Q}$, then $x = \dfrac{p}{q}$ for some $p, q \in \mathbb{N} \setminus \{0\}$ and $\gcd(p, q) = 1$. By the density of the irrational numbers in $\mathbb{R}$, we know that there exists a sequence $(x_n) \in (0, +\infty) \setminus \mathbb{Q}$ such that $\lim_{n\to\infty} x_n = x$. Then by definition, $f(x_n) = 0$ for all $n \in \mathbb{N}$, and so $\lim_{n\to\infty} f(x_n) = 0$. But we have that $f(x) = \dfrac{1}{p^2} \neq 0$. Hence, by the sequential criterion, f is not continuous at x.

Now suppose $c \in (0, +\infty) \setminus \mathbb{Q}$ and let $\varepsilon > 0$ be given. By the Archimedean Principle, there is an $m \in \mathbb{N}$ such that $\dfrac{1}{m^2} < \varepsilon$. Let S be the set defined in the following way:

$$S := \left\{ \frac{p}{q} \in \mathbb{Q} : \gcd(p, q) = 1, q \in \mathbb{N}, 1 \leq p \leq m \right\}$$

and let $\delta := \min\{|s - c| : s \in S\}$. Since $p \leq m$, one can show that $\delta > 0$. Now let $x \in (c - \delta, c + \delta)$. Then if x is irrational, we have

$$|f(x) - f(c)| = |0 - 0| = 0 < \varepsilon$$

as required. On the other hand, if x is rational, by construction it does not lie in S and therefore $x = \frac{p}{q}$ where $\gcd(p, q) = 1$ and $p > m$, so we have

$$|f(x) - f(c)| = \left|\frac{1}{p^2} - 0\right| = \frac{1}{p^2} \leq \frac{1}{m^2} < \varepsilon$$

as required. ◀

3. Let A be a subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ be a continuous function. Show that the set of c where $f(c) = 0$, is closed in A.

   Solution. Let $C := \{c \in A : f(c) = 0\}$. Let $(x_n) \subset C$ be a convergent sequence in C with limit $x := \lim_{n\to\infty} x_n \in A$. We want to show that $x \in C$. Since f is continuous, $0 = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x)$, so we have that $x \in C$ as required. ◀

4. Let A be a subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ be a continuous function. Show that

   $$f(\overline{E}) \subset \overline{f(E)}$$

   for all $E \subset A$. Show that it can be a proper subset by giving an example.

   Solution. Let $y \in f(\overline{E})$. Then $y = f(x)$ for some $x \in \overline{E}$. If $x \in E$, then $y = f(x) \in f(E) \subset \overline{f(E)}$ and we would be done. On the other hand, suppose $x \notin E$, then x is a limit point of E. So we can find a sequence $(x_n) \subset E \setminus \{x\}$ such that $\lim_{n\to\infty} x_n = x$. Then $(f(x_n))$ is a sequence in $f(E) \setminus \{y\}$ and by continuity, $(f(x_n))$ converges to $f(x) = y$. Hence, we see that y is a limit point of $f(E)$, that is, $y \in \overline{f(E)}$, and we are done.

   For the example of a proper subset, take $E := [1, +\infty)$ and $f(x) = \frac{1}{x}$. Then f is continuous on E and $f(E) = (0, 1]$. We have that $\overline{E} = [1, +\infty)$ and $f(\overline{(E)}) = (0, 1]$. But we see that $\overline{f(E)} = [0, 1] \neq (0, 1] = f(\overline{E})$. ◀

5. Let E be a closed subset in $\mathbb{R}$ and $f : E \to \mathbb{R}$ be a continuous function. Show that there exists $g : \mathbb{R} \to \mathbb{R}$ such that $g|_E = f$ and g is continuous. Is it true if E is not closed? Justify your answer.

   Solution. We will use the fact that an open set in $\mathbb{R}$ can be written as a union of open intervals. Since E is closed, $R \setminus E$ is open and hence we write $\mathbb{R} \setminus E = \bigcup_{\alpha \in A} I_\alpha = \bigcup_{\alpha \in A} (a_\alpha, b_\alpha)$ where $a_\alpha < b_\alpha$ are limit points of E. Then since E is closed, it contains its limit points, and hence f is defined at a_α, b_α for each $\alpha \in A$. So we define g in the following way:

   $$g(x) = \begin{cases} f(x), & x \in E \\ f(a_\alpha)\left(\frac{b_\alpha - x}{b_\alpha - a_\alpha}\right) + f(b_\alpha)\left(\frac{x - a_\alpha}{b_\alpha - a_\alpha}\right), & x \in (a_\alpha, b_\alpha) \text{ for some } \alpha \in A \end{cases}$$

that is, on (a_α, b_α), we define $g(x)$ as the linear interpolation between $f(a_\alpha)$ and $f(b_\alpha)$. Then by construction, $g : \mathbb{R} \to \mathbb{R}$ is continuous and $g|_E = f$.

To see that the result is not true when E is not closed, consider $f : (0, +\infty) \to \mathbb{R}$ by $f(x) = \dfrac{1}{x}$, which does not admit a continuous extension at 0. ◀

6. Show that $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{x}{x^2 + 2}$$

is continuous.

Solution. Let $c \in \mathbb{R}$ and let $\varepsilon > 0$ be given. Note that $x^2 + 2 \geq 2, c^2 + 2 \geq 2$, and we have

$$\left| \frac{x}{x^2+2} - \frac{c}{c^2+1} \right| = \left| \frac{2(x-c)}{(x^2+2)(c^2+2)} \right| \leq \frac{1}{2}|x - c|$$

and so taking $\delta := 2\varepsilon$, we see that if $|x-c| < \delta$, then $|f(x)-f(c)| < \varepsilon$ as required. ◀