THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I
Tutorial 10

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1. Using the $\varepsilon - \delta$ terminology, show that if $f : A \to B = f(A)$ and $g : B \to \mathbb{R}$ are two continuous functions, then $g \circ f : A \to \mathbb{R}$ is also a continuous function.

2. Show that $f : [0, +\infty) \to \mathbb{R}$ given by
$$f(x) = \frac{x}{x^2 + 1}$$
is uniformly continuous, using $\varepsilon - \delta$ terminology.

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x+1) = f(x)$ for all $x \in \mathbb{R}$, show that f is uniformly continuous. You might use the fact that a continuous function on compact domain is uniformly continuous.

1. Using the $\varepsilon - \delta$ terminology, show that if $f : A \to B = f(A)$ and $g : B \to \mathbb{R}$ are two continuous functions, then $g \circ f : A \to \mathbb{R}$ is also a continuous function.

Pf: Let $\varepsilon > 0$ be given. Let $c \in A$.

Then since g is cts on B, for any $\tilde{c} \in B$ $\exists\ \delta_0 > 0$ s.t. if $b \in B$, with $|b - \tilde{c}| < \delta_0$,

$|g(b) - g(\tilde{c})| < \varepsilon$.

Since f is cts, $\exists\ \delta_1 > 0$ s.t. for all $x \in A$ with $|x - c| < \delta_1$, we have that $|f(x) - f(c)| < \delta_0$. Then taking $b = f(x)$, $\tilde{c} = f(c)$ we see that $|g(f(x)) - g(f(c))| < \varepsilon$. //

2. Show that $f : [0, +\infty) \to \mathbb{R}$ given by

$$f(x) = \frac{x}{x^2 + 1}$$

is uniformly continuous, using $\varepsilon - \delta$ terminology.

Pf: Let $\varepsilon > 0$ be given. For $x, y \in [0, +\infty)$

$$\left| \frac{x}{x^2+1} - \frac{y}{y^2+1} \right| = \left| \frac{xy^2 + x - yx^2 - y}{(x^2+1)(y^2+1)} \right|$$

$$= \left| \frac{xy(y-x) + (x-y)}{(x^2+1)(y^2+1)} \right|$$

$$\leq |y-x| \underbrace{\left| \frac{xy}{(x^2+1)(y^2+1)} \right|}_{\text{consider either } 0 \leq x \leq 1,\ x > 1} + |y-x| \underbrace{\left| \frac{-1}{(x^2+1)(y^2+1)} \right|}_{x^2+1,\ y^2+1 \geq 1 \ \Rightarrow\ \leq 1}.$$

$$\leq 2|y-x|.$$

So taking $\delta = \frac{\varepsilon}{2}$ works.

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3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x+1) = f(x)$ for all $x \in \mathbb{R}$, show that f is uniformly continuous. You might use the fact that a continuous function on compact domain is uniformly continuous.

Pf: Let $\varepsilon > 0$ be given. $f|_{[0,2]}$ is uniformly cts, so $\exists\, \delta > 0$ s.t. if $x', y' \in [0,2]$ with $|y'-x'| < \delta$, then $|f(x') - f(y')| < \varepsilon$.

Let $x, y \in \mathbb{R}$ with $|x-y| < \delta' := \min\{\delta, 1\}$

So we write $x = \lfloor x \rfloor + (\underbrace{x - \lfloor x \rfloor}_{=:x'})$ where $\lfloor x \rfloor$ is the integral part of x.

and consider $y' = y - \lfloor x \rfloor$. Then

$$|x' - y'| = |x - \lfloor x \rfloor - (y - \lfloor x \rfloor)| = |x-y| < \delta.$$

$$x' = x - \lfloor x \rfloor \in [0,1] \subset [0,2].$$

$$|y'| = |y - \lfloor x \rfloor| \le |x - \lfloor x \rfloor| + |x-y| \le 1 + 1 = 2.$$ So $y' = y - \lfloor x \rfloor \in [0,2]$.

Since f is periodic with period 1, can show by induction that $f(x-n) = f(x)$ for any $n \in \mathbb{N}$.

Then $|f(x) - f(y)| = |f(x - \lfloor x \rfloor) - f(y - \lfloor x \rfloor)|$

$= |f(x') - f(y')| < \varepsilon$ by unif. cty of f on $[0,2]$.