THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I
Tutorial 8

Due Date: November 8, 2024

1. Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but that $|f|$ is continuous on $[0, 1]$.

2. (Exercises 5.2.5-5.2.6 of [BS11])

   (a) Let f, g be defined on $\mathbb{R}$ and let $c \in \mathbb{R}$. Suppose that $\lim_{x \to c} f = b$ and that g is continuous at b. Show that $\lim_{x \to c} g \circ f = g(b)$.

   (b) Does the conclusion hold if g is not continuous at b? Give an example showing otherwise.

3. Suppose $f : [0, 1] \to \mathbb{R}$ is a continuous function such that $f([0, 1]) \subset \mathbb{Q}$. Show that f is a constant function.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$f(m2^{-n}) = m2^{-n}$$

   for all $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Show that $f(x) = x$ for all $x \in \mathbb{R}$.

1. Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but that $|f|$ is continuous on $[0, 1]$.

Pf: $$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ -1 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \cap [0,1] \end{cases}$$

Clearly $|f| \equiv 1$ and so $|f|$ is cts.

let $c \in [0,1]$. Then $\exists (x_n) \subset \mathbb{Q} \cap [0,1]$, s.t. $x_n \to c$, $x_n \neq c$,

$\exists (y_n) \subset \mathbb{R}\backslash\mathbb{Q} \cap [0,1]$ s.t. $y_n \to c$, $y_n \neq c$

by density of $\mathbb{Q}$ in $\mathbb{R}$ and $\mathbb{R}\backslash\mathbb{Q}$ in $\mathbb{R}$.

But $f(x_n) = 1 \neq -1 = f(y_n)$.

So by sequential criterion for continuity, f is not cts at c. //

2. (Exercises 5.2.5-5.2.6 of [BS11])

   (a) Let f, g be defined on $\mathbb{R}$ and let $c \in \mathbb{R}$. Suppose that $\lim_{x \to c} f = b$ and that g is continuous at b. Show that $\lim_{x \to c} g \circ f = g(b)$.

   (b) Does the conclusion hold if g is not continuous at b? Give an example showing otherwise.

Pf: a) Let $\varepsilon > 0$ be given. Then since g is cts at b, $\exists\, \delta > 0$ s.t. for all $|x-b| < \delta$, we have $|g(x) - g(b)| < \varepsilon$.

Then since $\lim_{x \to c} f = b$, for $\varepsilon' = \delta$, there is a $\delta' > 0$ s.t. if $0 < |x-c| < \delta'$ we have that $|f(x) - b| < \varepsilon' = \delta$.

So chaining these together, we have that if $0 < |x-c| < \delta'$,

$$|f(x) - b| < \varepsilon' = \delta \implies |g(f(x)) - g(b)| < \varepsilon.$$

$$\shortparallel$$

$$|(g \circ f)(x) - g(b)| < \varepsilon.$$

b) Take $g(x) = \begin{cases} 0, & x = 1 \\ 2, & x \neq 1. \end{cases}$

Then g is not cts at 1. Set $f(x) = x + 1$ for all $x \in \mathbb{R}$.

Then $\lim_{x \to 0} f = 1$.

$(g \circ f)(0) = g(f(0)) = g(1) = 0$.

Let (x_n) be any sequence in $\mathbb{R}$ s.t. $x_n \neq 0$, and $x_n \to 0$.

$$(g \circ f)(x_n) = g(f(x_n)) = g(\underbrace{x_n + 1}_{\neq 1}) = 2.$$

So clearly $\lim_{x \to c} (g \circ f)(x) \neq (g \circ f)(0)$.

3. Suppose $f : [0,1] \to \mathbb{R}$ is a continuous function such that $f([0,1]) \subset \mathbb{Q}$. Show that f is a constant function.

Pf: Sps f is not constant. Then wlog, $\exists\, x_1 < x_2 \in [0,1]$, s.t. $f(x_1) < f(x_2)$. By density of irrationals in $\mathbb{R}$, we can find an $\alpha \in \mathbb{R} \setminus \mathbb{Q} \cap [0,1]$ s.t. $f(x_1) < \alpha < f(x_2)$.

Since f is cts, by intermediate value theorem, $\exists\, c \in [0,1]$ s.t. $f(c) = \alpha$, contradiction. $\cancel{c}$.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$f(m2^{-n}) = m2^{-n}$$

for all $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Show that $f(x) = x$ for all $x \in \mathbb{R}$.

Pf: First show $S = \{m2^{-n},\ m \in \mathbb{Z},\ n \in \mathbb{N}\}$ is dense in $\mathbb{R}$.

Let $x < y \in \mathbb{R}$. WTS $\exists s \in S$ s.t. $x < s < y$.

Since $y - x > 0$, by AP, $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{2^n} < \frac{1}{n} < y - x$

So $2^n y - 2^n x > 1$. So $\exists\, m \in \mathbb{Z}$ s.t.

$$2^n x < m < 2^n y.$$

So taking $s = m2^{-n}$, we have $x < s < y$.

So S is dense in $\mathbb{R}$.

Now show $f(x) = x$ for all $x \in \mathbb{R}$. By density of S in $\mathbb{R}$, for any $n \in \mathbb{N}$, there is an $x_n \in S$ s.t.

$$x < x_n < x + \frac{1}{n}.$$

Clearly $x_n \neq x$ for all $n \in \mathbb{N}$ and $x_n \to x$.

Then by sequential criterion for continuity,

$$f(x) = f\left(\lim_{n \to \infty} x_n\right) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n = x.$$

//.