MATH2050B Mathematical Analysis I

Suggested solution to HW 5

(1) Show that $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = \frac{1}{x^2+1}$$
is uniformly continuous by using ε-δ terminology.

Solution. For $z \in \mathbb{R}$, we have $(|z|-1)^2 \geq 0$ which implies that $z^2+1 \geq 2|z|$. Thus, for any $x, y \in \mathbb{R}$,

$$\begin{aligned}
|f(x)-f(y)| &= \frac{|x+y||x-y|}{(x^2+1)(y^2+1)} \\
&\leq \left(\frac{|x|}{(x^2+1)(y^2+1)} + \frac{|y|}{(x^2+1)(y^2+1)} \right) |x-y| \\
&\leq \left(\frac{1}{2(y^2+1)} + \frac{1}{2(x^2+1)} \right) |x-y| \\
&\leq \left(\frac{1}{2} + \frac{1}{2} \right) |x-y| = |x-y|.
\end{aligned}$$

Let $\varepsilon > 0$. Take $\delta = \varepsilon$. Now, if $x, y \in \mathbb{R}$ and $|x-y| < \delta$, we have

$$|f(x)-f(y)| \leq |x-y| < \delta = \varepsilon.$$

Hence f is uniformly continuous. □

(2) Suppose $f : [0, +\infty) \to \mathbb{R}$ is a continuous function such that $f|_{[a,+\infty)}$ is uniformly continuous for some $a > 0$. Show that f is uniformly continuous.

Solution. Let $\varepsilon > 0$. Since $f|_{[0,a+1]}$ is continuous, it is also uniformly continuous by the Uniform Continuity Theorem. So there exists $\delta_1 > 0$ such that

$$|f(u)-f(v)| < \varepsilon \quad \text{whenever } u, v \in [0, a+1] \text{ with } |u-v| < \delta_1.$$

On the other hand, since $f|_{[a,+\infty)}$ is uniformly continuous, there $\delta_2 > 0$ such that

$$|f(u)-f(v)| < \varepsilon \quad \text{whenever } u, v \in [a, +\infty) \text{ with } |u-v| < \delta_2.$$

Take $\delta := \{\delta_1, \delta_2, 1\}$. Now, if $u, v \in [0, +\infty)$ and $|u-v| < \delta$, then either both $u, v \in [a, +\infty)$ and $|u-v| < \delta \leq \delta_2$, so that $|f(u)-f(v)| < \varepsilon$; or one of $u, v \in [0, a]$, and hence both $u, v \in [0, a+1]$ (since $|u-v| \leq 1$), so that $|f(u)-f(v)| < \varepsilon$ because $|u-v| < \delta \leq \delta_1$.

□

(3) If $f, g : \mathbb{R} \to \mathbb{R}$ are two uniformly continuous functions, show that $f \circ g$ is also uniformly continuous.

Solution. Let $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta_1 > 0$ such that

$$|f(u) - f(v)| < \varepsilon \quad \text{whenever } u, v \in \mathbb{R} \text{ with } |u - v| < \delta_1.$$

Since g is uniformly continuous, there exists $\delta_2 > 0$ such that

$$|g(x) - g(y)| < \delta_1 \quad \text{whenever } x, y \in \mathbb{R} \text{ with } |x - y| < \delta_2.$$

Now, if $x, y \in \mathbb{R}$ and $|x - y| < \delta_2$, then $|g(x) - g(y)| < \delta_1$ and hence

$$|(f \circ g)(x) - (f \circ g)(y)| = |f(g(x)) - f(g(y))| < \varepsilon.$$

Therefore $f \circ g$ is uniformly continuous. □

(4) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\lim_{x \to +\infty} f(x) = L_1, \quad \lim_{x \to -\infty} f(x) = L_2$$

for some L_i. If $f(0) > \max\{L_1, L_2\}$, show that there exists $\bar{x} \in \mathbb{R}$ such that $f(\bar{x}) \geq f(x)$ for all $x \in \mathbb{R}$.

Solution. Since $\lim\limits_{x \to +\infty} f(x) = L_1$, $\lim\limits_{x \to -\infty} f(x) = L_2$, there are $\alpha_1 > 0$ and $\alpha_2 < 0$ such that

$$|f(x) - L_1| < \varepsilon_1 := f(0) - L_1 \quad \text{for all } x > \alpha_1,$$

and

$$|f(x) - L_2| < \varepsilon_2 := f(0) - L_2 \quad \text{for all } x < \alpha_2.$$

Thus $f(x) < f(0)$ if $x \in \mathbb{R} \backslash [\alpha_2, \alpha_1]$.

On the other hand, since $f|_{[\alpha_2, \alpha_1]}$ is continuous, it follows from the Maximum-Minimum Theorem that there exists $\bar{x} \in [\alpha_2, \alpha_1]$ such that $f(\bar{x}) \geq f(x)$ for all $x \in [\alpha_2, \alpha_1]$.

Note that $f(\bar{x}) \geq f(0)$ since $0 \in [\alpha_2, \alpha_1]$. Combining the inequalities above, we have $f(\bar{x}) \geq f(x)$ for all $x \in \mathbb{R}$. □

(5) Let A be a compact set in $\mathbb{R}$. Suppose $f : A \to \mathbb{R}$ is a real valued function such that for any $\varepsilon > 0$, there is a polynomial g_ε such that $\sup_A |f(x) - g_\varepsilon(x)| < \varepsilon$. Show that f is uniformly continuous.

Solution. Let $\varepsilon > 0$. By the assumption, there exists a polynomial g_ε such that $\sup_A |f(x) - g_\varepsilon(x)| < \varepsilon/3$. As a polynomial, g_ε is continuous on $\mathbb{R}$. Since A is a compact set in $\mathbb{R}$, g_ε is also uniformly continuous on A. Then there exists $\delta > 0$ such that

$$|g_\varepsilon(u) - g_\varepsilon(v)| < \varepsilon/3 \quad \text{whenever } u, v \in A \text{ with } |u - v| < \delta.$$

Now, if $u, v \in A$ and $|u - v| < \delta$, we have

$$\begin{aligned} |f(u) - f(v)| &\leq |f(u) - g_\varepsilon(u)| + |g_\varepsilon(u) - g_\varepsilon(v)| + |g_\varepsilon(v) - f(v)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Therefore f is uniformly continuous. □

(6) Suppose $f : (0, 1] \to \mathbb{R}$ is a bounded continuous function. Show that the function given by $g(x) = xf(x)$ is uniformly continuous on $(0, 1)$.

Solution. Since f is bounded, there exists $M > 0$ such that $|f(x)| \leq M$ for $x \in (0, 1]$. Let $\varepsilon > 0$. Fix $c \in (0, 1)$ such that $c < \frac{\varepsilon}{2M}$. Since $f|_{[c,1]}$ is continuous by the assumption, it is also uniformly continuous by the Uniform Continuity Theorem. Then there exists $\delta > 0$ such that

$$|f(u) - f(v)| < \varepsilon/2 \quad \text{whenever } u, v \in [c, 1] \text{ with } |u - v| < \delta.$$

Take $\delta' := \{\delta, \frac{\varepsilon}{2M}\}$. Now, if $u, v \in (0, 1)$ and $|u - v| < \delta'$, then either both $u, v \in [c, 1]$, so that

$$|uf(u) - vf(v)| \leq |u||f(u) - f(v)| + |f(v)||u - v| < 1 \cdot \frac{\varepsilon}{2} + M \cdot \frac{\varepsilon}{2M} = \varepsilon;$$

or at least one (say u) of u, v belongs to $(0, c)$, so that

$$|uf(u) - vf(v)| \leq |u||f(u) - f(v)| + |f(v)||u - v| < c \cdot 2M + M \cdot \frac{\varepsilon}{2M} \leq \varepsilon.$$

Therefore g is uniformly continuous on $(0, 1)$.

Alternative solution:

Since f is bounded, it follows from Squeeze Theorem that $\lim_{x \to 0^+} xf(x) = 0$. Thus g can be extended to a continuous function on $[0, 1]$ by defining $g(0) = 0$. By the Uniform Continuity Theorem, g is uniformly continuous on $[0, 1]$. In particular, g is uniformly continuous on $(0, 1)$. □