MATH2050B Mathematical Analysis I

Suggested solution to HW 4

(1) By using ε-δ terminology, show that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{x^2}{x^2 + 2x + 2}$$

is continuous.

Solution. Note that, for $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| = \frac{2|x - y||xy + x + y|}{((x + 1)^2 + 1)\,((y + 1)^2 + 1)} \leq 2|x - y||xy + x + y|.$$

In particular, for any $c \in \mathbb{R}$ and for any $\varepsilon > 0$, we take $\delta := \min\{\frac{\varepsilon}{2(c^2+3|c|+1)}, 1\}$ so that if $|x - c| < \delta$, then

$$|xc + x + c| \leq |x - c||c| + |c|^2 + |x - c| + 2|c| \leq c^2 + 3|c| + 1,$$

and hence

$$|f(x) - f(c)| \leq 2|x - c||xc + x + c| < 2(c^2 + 3|c| + 1)\delta \leq \varepsilon.$$

Thus f is continuous at c. Since this is true for any $c \in \mathbb{R}$, f is continuous on $\mathbb{R}$. □

(2) Let $f : (0, +\infty) \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} \dfrac{1}{m^2} & \text{if } x = \dfrac{m}{n}, \gcd(m, n) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Determine the set of c where f is continuous at c.

Solution. We will show that $\lim_{x\to c} f(x) = 0$ for any $c > 0$. So f is continuous at every irrational points in $(0, +\infty)$, and discontinuous elsewhere.

Fix $c > 0$. Let $\varepsilon > 0$ be given. Choose $k \in \mathbb{N}$ so that $1/k^2 < \varepsilon$. Since only finitely many points in the set

$$S_k := \left\{\frac{m}{n} : m, n \in \mathbb{N}, 1 \leq m \leq k\right\}$$

lie in $(\frac{c}{2}, \frac{3c}{2})$ (since $\frac{c}{2} < \frac{m}{n} < \frac{3c}{2}$, $1 \leq m \leq k \implies 1 \leq n \leq \frac{2k}{c}$), we can choose a $\delta > 0$ so small such that

$$S_k \cap (c - \delta, c + \delta)\backslash\{c\} = \emptyset.$$

Now if $0 < |x - c| < \delta$, then either $x \in \mathbb{R}\backslash\mathbb{Q}$ so that $|f(x)| = 0 < \varepsilon$;

or $x = \frac{m}{n}$ for some $m, n \in \mathbb{N}$ with $m > k$ so that $|f(x)| \leq \dfrac{1}{(k+1)^2} < \varepsilon$.

Hence $\lim_{x\to c} f(x) = 0$ for any $c > 0$. □

(3) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that for some $K > 0$, we have

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in \mathbb{R}$. Show that f is continuous. Is the converse true? Justify your answer.

Solution. The function is in fact uniformly continuous whence continuous. Indeed, for any $\varepsilon > 0$, we can take $\delta := \varepsilon/K$, so that if $|x - y| < \delta$, then

$$|f(x) - f(y)| \leq K|x - y| < K\delta = \varepsilon.$$

The converse is not true. Consider the function $f(x) = \sqrt{|x|}$. Clearly it is continuous on $\mathbb{R}$. However

$$\frac{|f(1/n) - f(0)|}{|1/n - 0|} = \frac{1/\sqrt{n}}{n} = \sqrt{n} \to +\infty \qquad \text{as } n \to \infty.$$

So no K satisfying the given condition can exist. □

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = 0$ for all x in form of $m2^{-n}$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Show that f is identically zero if f is continuous.

Solution. In view of Sequential Criterion for Continuity, it suffices to show that for any $c \in \mathbb{R}$, there is a sequence of the form $m2^{-n}$ that converges to c.

Indeed, for any $\varepsilon > 0$, choose $n \in \mathbb{N}$ such that $2^n\varepsilon > 1$. Then there must be an integer m between $2^n c$ and $2^n(c + \varepsilon)$ (since $|2^n(c + \varepsilon) - 2^n c| > 1$). So $c < m2^{-n} < c + \varepsilon$. As $\varepsilon > 0$ is arbitrary, such sequence above must exist. □

(5) Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions. Show that $h(x) = \max\{f(x), g(x)\}$ is also a continuous function.

Solution. The result follows immediately from Theorem 5.2.1, Theorem 5.2.4 and the fact that

$$h(x) = \max\{f(x), g(x)\} = \frac{f(x) + g(x) + |f(x) - g(x)|}{2}.$$

□