MATH2050B Mathematical Analysis I

Suggested solution to HW 3

(1) If $x_1 < x_2$ are some real numbers and $x_n = \frac{1}{4}x_{n-1} + \frac{3}{4}x_{n-2}$ for $n > 2$. Show that $\{x_n\}_{n=1}^{\infty}$ is convergent and find the limit.

Solution. Note that, for $n \in \mathbb{N}$,

$$x_{n+2} - x_{n+1} = \frac{1}{4}x_{n+1} + \frac{3}{4}x_n - x_{n+1} = -\frac{3}{4}(x_{n+1} - x_n),$$

and thus

$$x_{n+2} - x_{n+1} - \frac{3}{4}(x_{n+1} - x_n) = \cdots = (-\frac{3}{4})^n(x_2 - x_1).$$

Summing up the expression, we have

$$x_{n+2} - x_1 = \sum_{k=0}^{n}(x_{k+2} - x_{k+1}) = (x_2 - x_1)\sum_{k=0}^{n}(-\frac{3}{4})^n = (x_2 - x_1)\frac{1 - (-\frac{3}{4})^{n+1}}{1 - (-\frac{3}{4})}.$$

Therefore, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left\{x_1 + \frac{4}{7}(x_2 - x_1)[1 - (-\frac{3}{4})^{n+1}]\right\} = \frac{3}{7}x_1 + \frac{4}{7}x_2.$ □

(2) Let $x_1 = 1$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$. for all $n \in \mathbb{N}$. Show that the sequence is convergent and find the limit.

Solution. It is easy to see by induction that $\{x_n\}$ is a constant sequence of 1's. Clearly $\{x_n\}$ is convergent and the limit is 1. □

(3) Suppose all subsequence of (x_n) has a sub-sequence converging to 0. Show that $x_n \to 0$ as $n \to +\infty$.

Solution. Suppose $x_n \not\to 0$ as $n \to +\infty$. By Theorem 3.4.4, there exists $\varepsilon_0 > 0$ and a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k} - 0| \geq \varepsilon_0$ for all $k \in \mathbb{N}$. The assumption then implies that (x_{n_k}) has a further subsequence $(x_{n_{k_l}})$ converging to 0. This contradicts the condition $|x_{n_{k_l}} - 0| \geq \varepsilon_0$ for all $l \in \mathbb{N}$. Therefore $x_n \to 0$ as $n \to +\infty$. □

(4) Suppose (x_n) is a sequence of positive real numbers. Show that

$$\limsup_{n\to+\infty} x_n^{1/n} \leq \limsup_{n\to+\infty} \frac{x_{n+1}}{x_n}$$

provided that the limsup on the right hand side exists. Show that we cannot improve $\leq$ to $=$ by giving an example.

Solution. Let $\alpha > \limsup_{n\to+\infty} \frac{x_{n+1}}{x_n} = \inf_{n\geq 1}\left(\sup_{m\geq n} \frac{x_{m+1}}{x_m}\right).$

Then there exists $n_0 \in \mathbb{N}$ such that $\frac{x_{m+1}}{x_m} < \alpha$ for all $m \geq n_0$. Hence, for $m \geq n_0 + 1$,

$$\frac{x_m}{x_{n_0}} = \frac{x_{n_0+1}}{x_{n_0}} \cdot \frac{x_{n_0+2}}{x_{n_0+1}} \cdots \frac{x_m}{x_{m-1}} < \alpha^{m-n_0},$$

so that

$$\sqrt[m]{x_m} < \alpha^{1-\frac{n_0}{m}} x_{n_0}^{\frac{1}{m}} = \alpha \cdot \sqrt[m]{C},$$

where $C = x_{n_0}\alpha^{-n_0}$. Since $\lim\limits_{m\to+\infty} \sqrt[m]{C} = 1$, we have

$$\limsup_{m\to+\infty} \sqrt[m]{x_m} \leq \limsup_{m\to+\infty}(\alpha \cdot \sqrt[m]{C}) = \lim_{m\to+\infty}(\alpha \cdot \sqrt[m]{C}) = \alpha.$$

As $\alpha > \limsup\limits_{n\to+\infty} \frac{x_{n+1}}{x_n}$ is arbitrary, we have

$$\limsup_{n+\infty} \sqrt[n]{x_n} \leq \limsup_{n+\infty} \frac{x_{n+1}}{x_n}.$$

The inequality could be strict. For example, consider $x_n = 2 + (-1)^n$ for $n \in \mathbb{N}$. Then $\limsup\limits_{n\to+\infty} \sqrt[n]{x_n} = \lim\limits_{n\to+\infty} \sqrt[n]{x_n} = 1$ since $1 \leq \sqrt[n]{x_n} \leq \sqrt[n]{3}$ for all $n \in \mathbb{N}$. However, $\limsup\limits_{n\to+\infty} \frac{x_{n+1}}{x_n} = 3$ since $\frac{x_{n+1}}{x_n}$ is 3 when n is odd, and is 1/3 when n is even. □

(5) Suppose $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence such that x_n is an integer for any $n \in \mathbb{N}$. Show that there is N such that x_n is a constant for $n > N$.

Solution. Let $\varepsilon = 1/2$. By the definition of a Cauchy sequence, there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon = \frac{1}{2} \qquad \text{for } n, m \geq N.$$

In particular,

$$|x_n - x_N| < \frac{1}{2} \qquad \text{for } n \geq N.$$

Since x_n is an integer for any $n \in \mathbb{N}$, we must have $x_n = x_N$, that is a constant, for $n \geq N$. □

(6) Let $p \in \mathbb{N}$ be fixed. Construct an example of (x_n) which is not Cauchy but satisfies $|x_{n+p} - x_n| \to 0$ as $n \to +\infty$.

Solution. Consider the sequence (x_n) given by

$$x_n = \begin{cases} 1 & \text{if } n \text{ is a multiple of } p \\ 0 & \text{otherwise.} \end{cases}$$

Then $|x_{n+p} - x_n| = 0$ for all $n \in \mathbb{N}$. However, (x_n) is not Cauchy as it is divergent.

In fact, we can give an example of (x_n) does not depend on p. For example, $x_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is divergent hence not Cauchy. However, for any $p \in \mathbb{N}$,

$$|x_{n+p} - x_n| = \frac{1}{n+1} + \cdots + \frac{1}{n+p} \leq \frac{p}{n+1} \to 0 \quad \text{as } n \to +\infty.$$

□