MATH2050B Mathematical Analysis I

Suggested solution to HW 2

(1) Use the ε-N terminology, show the followings:

(a) $\lim_{n\to+\infty} \frac{n^3+2n+1}{n^3-2} = 1.$

(b) $\lim_{n\to+\infty} n^2 3^{-n} = 0.$

Solution. (a) Note that, for $n \geq 2$,

$$\left|\frac{n^3+2n+1}{n^3-2} - 1\right| = \left|\frac{2n+3}{n^3-2}\right| \leq \frac{2n+3n}{\frac{1}{2}n^3} = \frac{10}{n^2}.$$

Let $\varepsilon > 0$. By Archimedean Property, there is $N \in \mathbb{N}$ such that $N > \max\{2, 10/\varepsilon\}$. Now, if $n \geq N$, we have

$$\left|\frac{n^3+2n+1}{n^3-2} - 1\right| \leq \frac{10}{n^2} \leq \frac{10}{n} \leq \frac{10}{N} < \varepsilon.$$

Therefore $\lim_{n\to+\infty} \frac{n^3+2n+1}{n^3-2} = 1.$

(b) Note that, for $n \geq 3$, the Binomial Theorem yields

$$3^n = (1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k \geq \frac{n(n-1)(n-2)}{6} \cdot 2^3,$$

and so

$$0 \leq n^2 3^{-n} \leq \frac{6n}{8(n-1)(n-2)} \leq \frac{6n}{4n(n-2)} \leq \frac{2}{n-2}.$$

Let $\varepsilon > 0$. By Archimedean Property, there is $N \in \mathbb{N}$ such that $N > \max\{3, 2/\varepsilon + 2\}$. Now, if $n \geq N$, we have

$$\left|n^2 3^{-n} - 0\right| = n^2 3^{-n} \leq \frac{2}{n-2} \leq \frac{2}{N-2} < \varepsilon.$$

Therefore $\lim_{n\to+\infty} n^2 3^{-n} = 0.$

□

(2) Suppose (x_n) is a sequence of real numbers such that $x_n \to x$ for some $x \in \mathbb{R}$.

(a) If $x_n \in [a, b]$ for some a, b, show that $x \in [a, b]$.

(b) If $x \in (a, b)$, show that there exists N such that $x_n \in (a, b)$ for all $n > N$.

Solution. (a) Suppose on the contrary that $x < a$. Take $\varepsilon_0 = \frac{a-x}{2} > 0$. Since $x_n \to x$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - x| < \varepsilon_0$, and hence

$$x_n < x + \varepsilon_0 = x + \frac{a-x}{2} = \frac{x+a}{2} < a.$$

This contradicts the assumption that $x_n \in [a, b]$. Thus we must have $x \geq a$. Similarly, we can show that $x \leq b$.

(b) We are essentially repeating the argument in (a). Pick $\varepsilon_0 > 0$ such that $(x - \varepsilon_0, x + \varepsilon_0) \subset (a, b)$ (for example, one may take $\varepsilon_0 = \min\{x - a, b - x\}/2$). Since $x_n \to x$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - x| < \varepsilon_0$, and hence

$$x_n < x + \varepsilon_0 \leq x + \frac{b - x}{2} = \frac{b + x}{2} < b,$$

and

$$x_n > x - \varepsilon_0 \geq x - \frac{x - a}{2} = \frac{a + x}{2} > a.$$

Therefore, $x_n \in (a, b)$ for all $n \geq N$.

□

(3) Show that if $z_n = (a^n + b^n)^{1/n}$ for some distinct $a, b > 0$, then $z_n \to \max\{a, b\}$

Solution. Without loss of generality, we assume that $0 < a < b$. Then

$$b \leq z_n \leq (b^n + b^n)^{1/n} = b \cdot 2^{1/n} \qquad \text{for } n \in \mathbb{N}.$$

By Example 3.1.11(c) in the textbook, we have $\lim_{n \to +\infty} b \cdot 2^{1/n} = b \cdot 1 = b$. The Squeeze Theorem now implies that $\lim_{n \to +\infty} z_n = b = \max\{a, b\}$. □

(4) Suppose (x_n) is a sequence of positive real numbers such that $x_n^{1/n} \to L$ for some $L \in [0, 1)$. Show that $x_n \to 0$ as $n \to +\infty$. What if $L = 1$, what can you conclude? Justify your answer by either proving this or giving a counter-example.

Solution. If $L \in [0, 1)$, choose r such that $L < r < 1$. Since $x_n^{1/n} \to L$, there is $N \in \mathbb{N}$ such that

$$0 < x_n^{1/n} < r \qquad \text{for } n \geq N.$$

Thus

$$0 < x_n < r^n \qquad \text{for } n \geq N.$$

Note that $\lim_{n \to +\infty} r^n = 0$ since $0 < r < 1$. By the Squeeze Theorem, $\lim_{n \to +\infty} x_n = 0$.

If $L = 1$, then (x_n) may or may not converge.

For example, if $x_n = 1$ for all $n \in \mathbb{N}$, then $\lim_{n \to \infty} x_n^{1/n} = 1 = \lim_{n \to \infty} x_n$.

On the other hand, consider $x_n = 2 + (-1)^n$ for $n \in \mathbb{N}$. Clearly (x_n) is divergent as it oscillates between 1 and 3. Since

$$1 \leq x_n^{1/n} \leq 3^{1/n} \qquad \text{for } n \in \mathbb{N},$$

and $\lim_{n \to \infty} 3^{1/n} = 1$, we have $\lim_{n \to \infty} x_n^{1/n} = 1$ by the Squeeze Theorem. □