MATH2050B Mathematical Analysis I

Suggested solution to HW 1

(1) Using the Axioms to show that for all $a, b \in \mathbb{R}$,

$$(-a)^2 = a^2 \quad \text{and} \quad (a + (-b))^2 = a^2 + (-2ab) + b^2.$$

Solution. First we show that if $a + b = 0$, then $b = -a$ $(*)$. Indeed,

$$\begin{aligned} b &= b + 0 && \text{(by A3)} \\ &= b + (a + (-a)) && \text{(by A4)} \\ &= (a + b) + (-a) && \text{(by A1, A2)} \\ &= 0 + (-a) && \text{(by assumption)} \\ &= -a && \text{(by A3).} \end{aligned}$$

Thus, we have $-a = (-1)a$ $(**)$ because

$$\begin{aligned} a + (-1)a &= 1 \cdot a + (-1)a && \text{(by M3)} \\ &= (1 + (-1)) \cdot a && \text{(by D)} \\ &= 0 \cdot a && \text{(by A4)} \\ &= a \cdot 0 && \text{(by M1)} \\ &= 0 && \text{(by Theorem 2.1.2(c)).} \end{aligned}$$

Hence, to show that $(-a)^2 = a^2$, it suffices to show that $(-a)^2 + (-a^2) = 0$. Now

$$\begin{aligned} (-a)^2 + (-a^2) &= (-a)^2 + (-1)a^2 && \text{(by (**))} \\ &= (-a)^2 + ((-1)a)a && \text{(by M1, M2)} \\ &= (-a)^2 + (-a)a && \text{(by (**))} \\ &= (-a)(-a + a) && \text{(by D)} \\ &= (-a) \cdot 0 && \text{(by A4)} \\ &= 0 && \text{(by Theorem 2.1.2(c)).} \end{aligned}$$

For the second equality,

$$\begin{aligned} (a + (-b))^2 &= a(a + (-b)) + (-b)(a + (-b) && \text{(by D)} \\ &= a^2 + a(-b) + (-b)a + (-b)^2 && \text{(by D)} \\ &= a^2 + a(-b) + a(-b) + b^2 && \text{(by M1, first equality)} \\ &= a^2 + a((-1)b) + a((-1)b) + b^2 && \text{(by (**))} \\ &= a^2 + (-1)(ab) + (-1)(ab) + b^2 && \text{(by M1, M2)} \\ &= a^2 + (-1)(2ab) + b^2 && \text{(by D)} \\ &= a^2 + (-2ab) + b^2 && \text{(by (**)).} \end{aligned}$$

□

(2) If $x > 0$, show that there exists $n \in \mathbb{N}$ such that $3^{-n} < x$.

Solution. By Bernoulli's Inequality, $3^n = (1+2)^n \geq 1 + 2n > n$ for all $n \in \mathbb{N}$.

If $x > 0$, the Archimedean property implies that there exists $n \in \mathbb{N}$ such that $1/n < x$, and hence $3^{-n} < 1/n < x$. □

(3) Show that if A, B are bounded subsets of $\mathbb{R}$, then

$$\sup(A + B) = \sup A + \sup B$$

where $A + B = \{a + b : a \in A, b \in B\}$. Do we have

$$\sup A \cdot \sup B = \sup(A \cdot B)$$

where $A \cdot B = \{ab : a \in A, b \in B\}$? Justify your answer.

Solution. We further assume that A and B are non-empty. Otherwise, the corresponding suprema do not exist.

By the Completeness Axiom of $\mathbb{R}$, both $\sup A$ and $\sup B$ exist. It is clear that $\sup A + \sup B$ is an upper bound of $A + B$. Indeed, for any $a \in A$, $b \in B$, we have $a \leq \sup A$, $b \leq \sup B$ and hence $a + b \leq \sup A + \sup B$.

Next we show that $\sup A + \sup B$ is the least upper bound of $A + B$. Let $\varepsilon > 0$. By Lemma 2.3.4, there are $a_0 \in A$ and $b_0 \in B$ such that $a_0 > \sup A - \frac{\varepsilon}{2}$ and $b_0 > \sup B - \frac{\varepsilon}{2}$. Hence $a_0 + b_0 > \sup A + \sup B - \varepsilon$. By Lemma 2.3.4 again, we have $\sup(A + B) = \sup A + \sup B$.

Consider $A = B = \{-1, 0\}$. Then $\sup A = \sup B = 0$, and so $\sup A \cdot \sup B = 0$. However, $A \cdot B = \{0, 1\}$, so that $\sup(A \cdot B) = 1 \neq 0 = \sup A \cdot \sup B$. □

(4) Let X be a non-empty set and $f, g : X \to \mathbb{R}$ be two real-valued functions with bounded ranges. Show that

$$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$

Give an example showing that the inequality can be a strict inequality.

Solution. Since f and g have bounded ranges, $A \coloneqq \sup\{f(x) : x \in X\}$ and $B \coloneqq \sup\{g(x) : x \in X\}$ exist by the Completeness Axiom of $\mathbb{R}$. For all $x \in X$, we have $f(x) \leq A$ and $g(x) \leq B$, and so

$$f(x) + g(x) \leq A + B.$$

Now $A + B$ is an upper bound of $\{f(x) + g(x) : x \in X\}$, and thus

$$\sup\{f(x) + g(x) : x \in X\} \leq A + B = \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$

Let $X = [-1, 1]$ and let $f, g : X \to \mathbb{R}$ be defined by $f(x) = -g(x) = x$. Then

$$\sup\{f(x) + g(x) : x \in X\} = 0 < 1 + 1 = \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$

□

(5) Show by using completeness that there is a unique $x \in \mathbb{R}$ so that $x > 0$ and $x^3 = 4$.

Solution. Let $S = \{s \in \mathbb{R} : s^3 < 4\}$. Then $S \neq \emptyset$ (for example $1 \in S$) and S is bounded above by 2 (for $s > 2 \implies s^3 > 8 > 4$). By the Completeness Axiom of $\mathbb{R}$, $x := \sup S$ exists. Clearly $x \geq 1 > 0$. We will show that $x^3 = 4$ by ruling out the other two possibilities: $x^3 < 4$ and $x^3 > 4$.

We will make use of the following elementary inequality: if $0 \leq a \leq b$, then

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1}) \leq (b-a)nb^{n-1}.$$

Suppose $x^3 < 4$. Take $\varepsilon = \frac{1}{2}\min\{\frac{4-x^3}{3(x+1)^2}, 1\} > 0$. Then

$$(x+\varepsilon)^3 - x^3 \leq 3\varepsilon(x+\varepsilon)^2 \leq \frac{(x+\varepsilon)^2}{2(x+1)^2}(4-x^3) < 4 - x^3,$$

and so $(x+\varepsilon)^3 < 4$. Since $x < x + \varepsilon$, this contradicts the fact that $x = \sup S$ is an upper bound of S.

Suppose $x^3 > 4$. Take $\varepsilon = \frac{x^3-4}{6x^2} \in (0, x)$. Then

$$x^3 - (x-\varepsilon)^3 \leq 3\varepsilon x^2 = \frac{1}{2}(x^3 - 4) < x^3 - 4,$$

and so $(x-\varepsilon)^3 > 4$. Now $x - \varepsilon$ is an upper bound of S since $s > x - \varepsilon \implies s^3 > (x-\varepsilon)^3 > 4$. Again this contradicts the fact that $x = \sup S$ is the least upper bound of S.

Thus, we must have $x^3 = 4$.

The uniqueness of such x is clear because $0 < u < v \implies u^3 < v^3$.

Therefore, there is a unique $x \in \mathbb{R}$ so that $x > 0$ and $x^3 = 4$.

$\square$